

Written Final and Entrance Exam - 1988

translated from: KoMaL, September 1988, pages 251-252

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These problems were given at the joint written final (érettségi) and entrance examination in mathematics in Hungary in 1988. Problems 1–5 were the same for all students; for problems 6–8, there was one version (A) recommended for students from academic high schools and another one (B) for students from vocational secondary schools. The points for each problem are shown in parentheses.

For all students

1. Solve the given system of equations over the real numbers.

$$\frac{4}{x} + \frac{5}{y} = 23, \quad \frac{5}{x} - \frac{4}{y} = -2$$

(9 points)

Solution: Let $u = \frac{1}{x}$ and $v = \frac{1}{y}$. Then $4u + 5v = 23$ and $5u - 4v = -2$. Multiplying the first equation by 4 and the second one by 5 and adding them, we get $41u = 82$, so $u = 2$ and then $5v = 23 - 8 = 15$, $v = 3$. So

$$x = \frac{1}{2}, \quad y = \frac{1}{3}$$

2. Solve the given equation over the real numbers.

$$\left(\frac{3}{4}\right)^{\log x} + \left(\frac{4}{3}\right)^{\log x} = \frac{25}{12}$$

(11 points)

Solution: We need $x > 0$. Let $t = \left(\frac{3}{4}\right)^{\log x} > 0$. Then $\left(\frac{4}{3}\right)^{\log x} = \frac{1}{t}$, so

$$t + \frac{1}{t} = \frac{25}{12}, \quad 12t^2 - 25t + 12 = 0, \quad (4t - 3)(3t - 4) = 0$$

If $t = \frac{3}{4}$, then $\log x = 1$ and $x = 10$. If $t = \frac{4}{3}$, then $\log x = -1$ and $x = \frac{1}{10}$. The solutions are $x = 10$ and $x = \frac{1}{10}$.

3. A rhombus with a 30° angle is drawn around a circle of area t . Express the area of the rhombus in terms of t . (12 points)

Solution: Let r be the radius of the circle, so $t = r^2\pi$. The height of the rhombus is the diameter of the circle, $2r$. If the side of the rhombus is a , then its height is $a \sin 30^\circ = \frac{a}{2}$, so $a = 4r$. The area of the rhombus is

$$a \cdot 2r = 8r^2 = \frac{8t}{\pi}$$

4. For which real numbers x are the following expressions defined?

a) $\sqrt{3 - \sqrt{x-2}}$ b) $\sqrt{x - \sqrt{x-2}}$ c) $\sqrt{1 - \tan^2 x}$

(13 points)

Solution: a) We need $x \geq 2$ and $\sqrt{x-2} \leq 3$, that is, $x-2 \leq 9$. The expression is defined for $2 \leq x \leq 11$.

b) We need $x \geq 2$ and $\sqrt{x-2} \leq x$. For $x \geq 2$ both sides are nonnegative, so this is equivalent to $x-2 \leq x^2$, that is, $x^2 - x + 2 \geq 0$. This is true for every x , because $x^2 - x + 2 = \left(x - \frac{1}{2}\right)^2 + \frac{7}{4} > 0$. The expression is defined for $x \geq 2$.

c) We need $\tan x$ to be defined and $\tan^2 x \leq 1$, that is, $-1 \leq \tan x \leq 1$. The expression is defined for

$$-\frac{\pi}{4} + k\pi \leq x \leq \frac{\pi}{4} + k\pi, \quad k \text{ is an integer}$$

5. In triangle ABC , the angle at vertex A is twice the angle at vertex B , twice the side BC is equal to three times the side AC , and the area of the triangle is 27 times the sine of the angle at vertex C . Find the angles and the sides of the triangle. (13 points)

Solution: Let $a = BC$, $b = AC$, $c = AB$, and let β be the angle at B , so the angle at A is 2β . By the Law of Sines,

$$\frac{a}{b} = \frac{\sin 2\beta}{\sin \beta} = 2 \cos \beta$$

and $\frac{a}{b} = \frac{3}{2}$, so $\cos \beta = \frac{3}{4}$. The area is $\frac{1}{2}ab \sin \gamma = 27 \sin \gamma$, so $ab = 54$. With $a = \frac{3}{2}b$ this gives $b^2 = 36$, $b = 6$, and $a = 9$. The angle at C is $\gamma = 180^\circ - 3\beta$, so again by the Law of Sines,

$$\frac{c}{b} = \frac{\sin 3\beta}{\sin \beta} = 3 - 4 \sin^2 \beta = 3 - 4 \cdot \frac{7}{16} = \frac{5}{4}$$

(using $\sin 3\beta = 3 \sin \beta - 4 \sin^3 \beta$), so $c = 7.5$. The sides are

$$BC = 9, \quad AC = 6, \quad AB = 7.5$$

and the angles are $\beta = \cos^{-1} \frac{3}{4} \approx 41.41^\circ$ at B , $2\beta \approx 82.82^\circ$ at A , and $180^\circ - 3\beta \approx 55.77^\circ$ at C .

A. Recommended for students from academic high schools

6. Find the equation of the circle whose center is on the x -axis, and which is tangent from the inside to the circle k_1 : $x^2 + y^2 + 4x + 6y - 87 = 0$, and tangent from the outside to the circle k_2 : $x^2 + y^2 - 20x - 12y + 111 = 0$. (13 points)

Solution: Completing the squares,

$$k_1: (x+2)^2 + (y+3)^2 = 100, \quad k_2: (x-10)^2 + (y-6)^2 = 25$$

so k_1 has center $O_1(-2, -3)$ and radius 10, and k_2 has center $O_2(10, 6)$ and radius 5. Let the center of the circle be K and its radius r . The circle is inside k_1 and touches it, so $O_1K = 10 - r$, and it touches k_2 from the outside, so $O_2K = 5 + r$. Then

$$O_1K + KO_2 = 15 = \sqrt{12^2 + 9^2} = O_1O_2$$

so K is on the segment O_1O_2 . (So k_1 and k_2 touch each other, and the circle we are looking for fits between them.)

The points of the segment are $(-2 + 12s, -3 + 9s)$ with $0 \leq s \leq 1$, and K is on the x -axis, so $s = \frac{1}{3}$ and $K = (2, 0)$. Then $r = 10 - O_1K = 10 - 5 = 5$, and $O_2K = 10 = 5 + r$ checks. The circle is

$$(x-2)^2 + y^2 = 25$$

7. The base of a pyramid is a square, and its lateral faces are congruent isosceles triangles. The height of the pyramid makes a 30° angle with the lateral faces. A cube is placed inside the pyramid so that four vertices of the cube are on the base of the pyramid, and the other four vertices are on the lateral faces, on the altitudes of the lateral faces drawn from the apex of the pyramid. Find the ratio of the volumes of the pyramid and the cube. (14 points)

Solution: Let the side of the base be 2 and let the base be centered at the origin with sides parallel to the axes. The altitudes of the lateral faces from the apex go from the apex to the midpoints $(\pm 1, 0)$, $(0, \pm 1)$ of the base edges. The height m of the pyramid makes 30° with the lateral faces, so the lateral faces make 60° with the base, and $m = 1 \cdot \tan 60^\circ = \sqrt{3}$.

The four upper vertices of the cube are on these four altitudes, so the cube is turned 45° compared to the base: seen from above, its vertices are on the lines $x = 0$ and $y = 0$ (which join the midpoints of opposite base edges), at distance $\frac{x}{\sqrt{2}}$ from the center, where x is the edge of the cube. The upper vertices are at height x , where the cross-section of the pyramid is a square of half-side $1 - \frac{x}{\sqrt{3}}$ (the pyramid shrinks linearly to the apex), and the upper vertices are the midpoints of the sides of this square. So

$$\frac{x}{\sqrt{2}} = 1 - \frac{x}{\sqrt{3}}, \quad x = \frac{1}{\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}}} = \frac{\sqrt{6}}{\sqrt{3} + \sqrt{2}} = \sqrt{6}(\sqrt{3} - \sqrt{2}) = 3\sqrt{2} - 2\sqrt{3} \approx 0.779$$

The volume of the pyramid is $\frac{1}{3} \cdot 4 \cdot \sqrt{3} = \frac{4\sqrt{3}}{3}$, and $\frac{1}{x^3} = \frac{(\sqrt{3} + \sqrt{2})^3}{6\sqrt{6}} = \frac{9\sqrt{3} + 11\sqrt{2}}{6\sqrt{6}}$. So the ratio of the volumes is

$$\frac{V_{\text{pyramid}}}{V_{\text{cube}}} = \frac{4\sqrt{3}}{3} \cdot \frac{9\sqrt{3} + 11\sqrt{2}}{6\sqrt{6}} = \frac{2(9\sqrt{3} + 11\sqrt{2})}{9\sqrt{2}} = \sqrt{6} + \frac{22}{9} \approx 4.894$$

8. Let p and q be positive integers with $q \geq p$. Prove that

$$\frac{q + \frac{1}{2}}{p - \frac{1}{2}} > \frac{1}{p} + \frac{1}{p+1} + \frac{1}{p+2} + \cdots + \frac{1}{q}$$

(15 points)

Solution: The right-hand side has $q - p + 1$ terms, and each of them is at most $\frac{1}{p}$, so the right-hand side is at most

$\frac{q - p + 1}{p}$. The left-hand side is greater than $\frac{q + \frac{1}{2}}{p}$, because $0 < p - \frac{1}{2} < p$. Finally, $q + \frac{1}{2} > q - p + 1$, because $p \geq 1 > \frac{1}{2}$. So

$$\frac{q + \frac{1}{2}}{p - \frac{1}{2}} > \frac{q + \frac{1}{2}}{p} > \frac{q - p + 1}{p} \geq \frac{1}{p} + \frac{1}{p+1} + \cdots + \frac{1}{q}$$

Remark: The inequality is true in a much stronger form too: the sum on the right is less than $\ln \frac{q + \frac{1}{2}}{p - \frac{1}{2}}$. This follows

from the fact that each term $\frac{1}{k}$ is less than the area under the graph of $y = \frac{1}{x}$ between $k - \frac{1}{2}$ and $k + \frac{1}{2}$, because the graph is convex.

B. Recommended for students from vocational secondary schools

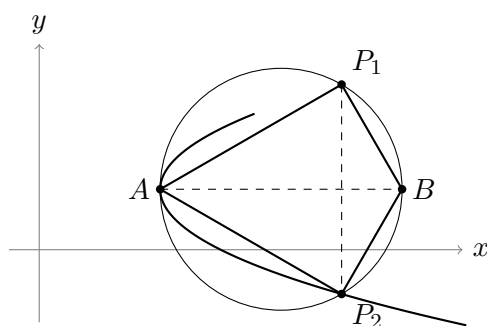
6. From the points P_1 and P_2 of the parabola $x - 2 = (y - 1)^2$, the segment with endpoints $A(2, 1)$ and $B(6, 1)$ is seen at a right angle. Find the area of the quadrilateral AP_1BP_2 . (13 points)

Solution: The points from which AB is seen at a right angle are on the circle with diameter AB (Thales' theorem), whose center is $(4, 1)$ and radius is 2: $(x - 4)^2 + (y - 1)^2 = 4$. Let $u = y - 1$; on the parabola $x = 2 + u^2$, so

$$(u^2 - 2)^2 + u^2 = 4, \quad u^4 - 3u^2 = 0$$

$u = 0$ gives the point A itself (the vertex of the parabola), and $u = \pm\sqrt{3}$ gives $P_1(5, 1 + \sqrt{3})$ and $P_2(5, 1 - \sqrt{3})$. The diagonals of the quadrilateral AP_1BP_2 are $AB = 4$ and $P_1P_2 = 2\sqrt{3}$, and they are perpendicular, so its area is

$$\frac{4 \cdot 2\sqrt{3}}{2} = 4\sqrt{3} \approx 6.93$$



7. A right circular cylinder has height 2 and radius 7. A square is inscribed in the cylinder so that its plane intersects the axis of the cylinder, and its vertices are on the circles of the base and of the top. Find the side of the square and the angle between the axis of the cylinder and the plane of the square. (14 points)

Solution: Two vertices of the square are on the base circle and two on the top circle, so two opposite sides of the square are horizontal chords of these circles. By symmetry, these chords are at the same distance d from the axis, on opposite sides of it. If the side of the square is s , then a chord gives $s^2 = 4(49 - d^2)$. A side going from the base to the top rises 2 units, while its projection to the base is perpendicular to the chords and has length $2d$, so $s^2 = 4d^2 + 4$. Then

$$4(49 - d^2) = 4d^2 + 4, \quad d^2 = 24, \quad s^2 = 100, \quad s = 10$$

The angle φ between the axis and the plane of the square is the angle between the axis and a side going from the base to the top (this side is the steepest line in the plane). So

$$\cos \varphi = \frac{2}{s} = \frac{1}{5}, \quad \varphi \approx 78.46^\circ$$

8. Solve the given inequality over the real numbers.

$$2\sqrt{|1-2x|} \geq 1+x$$

(15 points)

Solution: If $x < -1$, then the right-hand side is negative and the inequality is true. If $x \geq -1$, both sides are nonnegative, and we can square: $4|1-2x| \geq (1+x)^2$.

If $-1 \leq x \leq \frac{1}{2}$: $4-8x \geq x^2+2x+1$, that is, $x^2+10x-3 \leq 0$, so $-5-2\sqrt{7} \leq x \leq -5+2\sqrt{7}$. Together with the condition: $-1 \leq x \leq 2\sqrt{7}-5$ (note that $2\sqrt{7}-5 \approx 0.29 < \frac{1}{2}$).

If $x > \frac{1}{2}$: $8x-4 \geq x^2+2x+1$, that is, $x^2-6x+5 \leq 0$, so $1 \leq x \leq 5$.

The solution is

$$x \leq 2\sqrt{7}-5 \approx 0.29 \quad \text{or} \quad 1 \leq x \leq 5$$

Technical Universities Entrance Exam - 1988

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These problems were given at the written mathematics entrance examination of the technical universities and colleges in Hungary in 1988. Problems 1–5 were the same for all applicants; for problems 6–8, there was one version recommended for students from academic high schools and another one for students from vocational secondary schools. The points for each problem are shown in parentheses.

For all applicants

- Find the exact values of the given expressions without using approximate values.

$$A = 10^{1-\log 2} + 3^{\log_9 36}, \quad B = 125^{-\frac{1}{3}} \cdot \sqrt[3]{1000} \cdot \log_{\sin \frac{\pi}{6}} \sqrt{2}, \quad C = \sqrt{\log \tan 36^\circ + \log \tan 54^\circ}$$

(9 points)

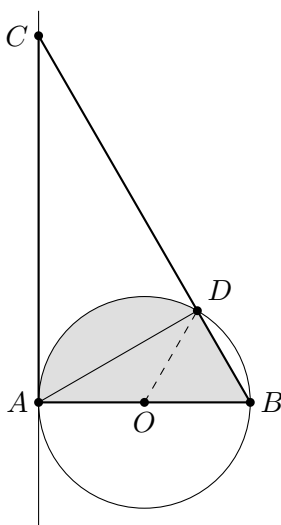
Solution: $10^{1-\log 2} = \frac{10}{10^{\log 2}} = \frac{10}{2} = 5$, and $3^{\log_9 36} = 3^{\log_3 6} = 6$ (because $\log_9 36 = \frac{\log_3 36}{\log_3 9} = \frac{2 \log_3 6}{2}$). So $A = 11$.

$125^{-\frac{1}{3}} = \frac{1}{5}$, $\sqrt[3]{1000} = 10$, and $\sin \frac{\pi}{6} = \frac{1}{2}$, $\log_{\frac{1}{2}} \sqrt{2} = -\frac{1}{2}$ (because $\left(\frac{1}{2}\right)^{-\frac{1}{2}} = \sqrt{2}$). So $B = \frac{1}{5} \cdot 10 \cdot \left(-\frac{1}{2}\right) = -1$.

$\log \tan 36^\circ + \log \tan 54^\circ = \log (\tan 36^\circ \tan 54^\circ) = \log 1 = 0$, because $\tan 54^\circ = \tan (90^\circ - 36^\circ) = \frac{1}{\tan 36^\circ}$. So $C = 0$.

- Let AB be a diameter of a circle of radius 1, and let C be the point on the tangent line to the circle at A for which the segment AC has length $2\sqrt{3}$. Find the area of the common part of the triangle ABC and the disk. (11 points)

Solution: The triangle ABC has a right angle at A , with $AB = 2$ and $AC = 2\sqrt{3}$, so $\tan \angle ABC = \sqrt{3}$ and $\angle ABC = 60^\circ$. Let D be the second intersection point of BC with the circle. Since AB is a diameter, $\angle ADB = 90^\circ$, so $BD = AB \cos 60^\circ = 1$ and $AD = AB \sin 60^\circ = \sqrt{3}$.



The common part consists of the triangle ABD and the circular segment cut off by the chord AD . The area of the triangle ABD is $\frac{1}{2} \cdot 1 \cdot \sqrt{3} = \frac{\sqrt{3}}{2}$. The central angle over the chord AD is $\angle AOD = 2\angle ABD = 120^\circ$, so the area of the segment is

$$\frac{1}{2} \left(\frac{2\pi}{3} - \sin 120^\circ \right) = \frac{\pi}{3} - \frac{\sqrt{3}}{4}$$

The area of the common part is

$$\frac{\sqrt{3}}{2} + \frac{\pi}{3} - \frac{\sqrt{3}}{4} = \frac{\pi}{3} + \frac{\sqrt{3}}{4} \approx 1.48$$

3. We build a large cube out of small cubes with unit edges. If we build the largest possible cube from the small cubes we have, then 107 small cubes are left over. If we want one more small cube along each edge, then we would need 62 more small cubes. How many small cubes do we have? (12 points)

Solution: If the largest cube has n small cubes along each edge, then the number of small cubes is

$$n^3 + 107 = (n + 1)^3 - 62, \quad 3n^2 + 3n + 1 = 169, \quad n^2 + n - 56 = 0$$

so $(n + 8)(n - 7) = 0$ and $n = 7$. We have $7^3 + 107 = 450$ small cubes. (Check: $8^3 - 62 = 512 - 62 = 450$.)

4. For which pairs of real numbers is the following system of equations true?

$$\begin{aligned} y - x &= 44 \\ \sqrt{\frac{6x}{x+y}} + \sqrt{\frac{x+y}{6x}} &= \frac{5}{2} \end{aligned}$$

(13 points)

Solution: Let $t = \sqrt{\frac{6x}{x+y}} > 0$. Then

$$t + \frac{1}{t} = \frac{5}{2}, \quad 2t^2 - 5t + 2 = 0, \quad t = 2 \text{ or } t = \frac{1}{2}$$

If $t = 2$: $\frac{6x}{x+y} = 4$, so $6x = 4x + 4y$, $x = 2y$. Then $y - 2y = 44$, so $y = -44$ and $x = -88$.

If $t = \frac{1}{2}$: $\frac{6x}{x+y} = \frac{1}{4}$, so $24x = x + y$, $y = 23x$. Then $22x = 44$, so $x = 2$ and $y = 46$.

Both pairs work (in the first one, $\frac{6x}{x+y} = \frac{-528}{-132} = 4$). The solutions are $(x, y) = (-88, -44)$ and $(2, 46)$.

5. The line $x + y = 30$ intersects the x -axis at A and the y -axis at B , and let O be the origin. Find the equation of the line e that is parallel to the line $x - 3y = 6$, intersects the segment OB at C and the segment AB at D , and for which the area of the quadrilateral $OADC$ is 234 units. (13 points)

Solution: $A = (30, 0)$ and $B = (0, 30)$. The line e is parallel to $x - 3y = 6$, so its equation is $x - 3y = -3k$ for some number k ; then $C = (0, k)$, with $0 < k < 30$. Solving $x + y = 30$ and $x - 3y = -3k$: $D = \left(\frac{90 - 3k}{4}, \frac{30 + 3k}{4}\right)$. The quadrilateral $OADC$ is made of the triangles OAD and ODC :

$$\frac{1}{2} \cdot 30 \cdot \frac{30 + 3k}{4} + \frac{1}{2} \cdot k \cdot \frac{90 - 3k}{4} = 234$$

Multiplying by 8:

$$900 + 90k + 90k - 3k^2 = 1872, \quad k^2 - 60k + 324 = 0, \quad (k - 6)(k - 54) = 0$$

Since $k < 30$, $k = 6$. Then $C = (0, 6)$, $D = (18, 12)$, and the equation of e is

$$x - 3y = -18$$

Recommended for students from academic high schools

6. For given real numbers p and q , find the first term and the difference of every arithmetic sequence in which the sum of the first four terms is p , and the quotient of the fourth term and the first term is q . (13 points)

Solution: Let the first term be $a \neq 0$ and the difference d . Then

$$4a + 6d = p, \quad \frac{a + 3d}{a} = q$$

The second equation gives $3d = (q - 1)a$, and then the first one becomes

$$4a + 2(q - 1)a = p, \quad 2(q + 1)a = p$$

If $q \neq -1$ and $p \neq 0$: there is exactly one such sequence,

$$a = \frac{p}{2(q + 1)}, \quad d = \frac{(q - 1)p}{6(q + 1)}$$

If $q \neq -1$ and $p = 0$: then $a = 0$, which is not allowed (the quotient is not defined), so there is no such sequence.

If $q = -1$ and $p \neq 0$: the equation $0 \cdot a = p$ has no solution, so there is no such sequence.

If $q = -1$ and $p = 0$: every $a \neq 0$ works, with $d = -\frac{2}{3}a$. (For example 3, 1, -1, -3.)

7. Solve the given equation, if p is a prime number and n is a positive integer.

$$1 + p + p^2 + \cdots + p^n = 2801$$

(14 points)

Solution: Subtracting 1:

$$p(1 + p + \cdots + p^{n-1}) = 2800 = 2^4 \cdot 5^2 \cdot 7$$

so p is 2, 5, or 7. If $p = 2$, then the left-hand side of the original equation is $2^{n+1} - 1$, which is never 2801, because 2802 is not a power of 2. If $p = 5$: $1 + 5 + 25 + 125 + 625 = 781$ and $781 + 3125 = 3906$, so the sum skips 2801. If $p = 7$: $1 + 7 + 49 + 343 + 2401 = 2801$. The only solution is $p = 7$, $n = 4$.

8. Consider four vertices of a unit cube such that no two of them are on the same edge of the cube. The tetrahedron determined by these four points is cut by planes parallel to two opposite edges of the tetrahedron. Considering all such planes, what are the largest possible perimeter and the largest possible area of the cross-section? (15 points)

Solution: The four vertices are pairwise endpoints of face diagonals of the cube, so they form a regular tetrahedron with edge $\sqrt{2}$. Let the plane be parallel to the opposite edges AB and CD . It cuts the edges AC , AD , BD , BC at K , L , M , N . If $\frac{AK}{AC} = \lambda$ with $0 < \lambda < 1$, then (by the intercept theorem) $KL \parallel CD$ with $KL = \lambda\sqrt{2}$, and $KN \parallel AB$ with $KN = (1 - \lambda)\sqrt{2}$, and similarly for the other two sides. So the cross-section is a parallelogram with sides $\lambda\sqrt{2}$ and $(1 - \lambda)\sqrt{2}$. Opposite edges of a regular tetrahedron are perpendicular (in the cube, AB and CD are diagonals of opposite faces in perpendicular directions), so the cross-section is a rectangle.

Its perimeter is $2(\lambda\sqrt{2} + (1 - \lambda)\sqrt{2}) = 2\sqrt{2}$, the same for every such plane, so its largest value is $2\sqrt{2}$.

Its area is $2\lambda(1 - \lambda)$, and by the inequality between the arithmetic and geometric means,

$$\lambda(1 - \lambda) \leq \left(\frac{\lambda + (1 - \lambda)}{2} \right)^2 = \frac{1}{4}$$

with equality for $\lambda = \frac{1}{2}$. So the largest area is $\frac{1}{2}$, for the plane through the midpoints of the edges: the cross-section is then a square with side $\frac{\sqrt{2}}{2}$.

Recommended for students from vocational secondary schools

6. The difference of the volumes of two right circular cylinders is 48π , and the difference of their lateral surface areas is 24π . The radius of the cylinder with the larger volume is 1 more than the radius of the other one, and its height is $\frac{4}{3}$ times the height of the other one. Find the heights and the radii of the cylinders. (13 points)

Solution: Let the smaller cylinder have radius r and height h ; then the larger one has radius $r + 1$ and height $\frac{4}{3}h$. So

$$\pi(r + 1)^2 \cdot \frac{4}{3}h - \pi r^2 h = 48\pi, \quad 2\pi(r + 1) \cdot \frac{4}{3}h - 2\pi r h = 24\pi$$

Multiplying by $\frac{3}{\pi}$ and by $\frac{3}{2\pi}$:

$$h(r^2 + 8r + 4) = 144, \quad h(r + 4) = 36$$

Dividing the first equation by the second one:

$$\frac{r^2 + 8r + 4}{r + 4} = 4, \quad r^2 + 4r - 12 = 0, \quad (r + 6)(r - 2) = 0$$

so $r = 2$ and $h = \frac{36}{6} = 6$. The smaller cylinder has radius 2 and height 6, the larger one has radius 3 and height 8. (Check: $72\pi - 24\pi = 48\pi$ and $48\pi - 24\pi = 24\pi$.)

7. Solve the given equation over the pairs of real numbers.

$$2 \cos^2 \frac{x^2 + 3y}{6} = 3^x + 3^{-x}$$

(14 points)

Solution: The left-hand side is at most 2. By the inequality between the arithmetic and geometric means, the right-hand side is at least $2\sqrt{3^x \cdot 3^{-x}} = 2$, with equality only if $3^x = 3^{-x}$, that is, $x = 0$. So both sides must be 2: $x = 0$ and $\cos^2 \frac{y}{2} = 1$, that is, $\frac{y}{2} = k\pi$. The solutions are

$$x = 0, \quad y = 2k\pi, \quad k \text{ is an integer}$$

8. In a geometric sequence of positive numbers, the sum of the first n terms is S , and the sum of the reciprocals of the first n terms is R . Express the product of the first n terms in terms of S , R , and n . (15 points)

Solution: Let the terms be $a, aq, \dots, aq^{n-1}$ with $a > 0, q > 0$. Then

$$R = \frac{1}{a} + \frac{1}{aq} + \dots + \frac{1}{aq^{n-1}} = \frac{q^{n-1} + q^{n-2} + \dots + 1}{aq^{n-1}} = \frac{S}{a^2 q^{n-1}}$$

because $q^{n-1} + \dots + 1 = \frac{S}{a}$. So $a^2 q^{n-1} = \frac{S}{R}$. The product of the terms is

$$a^n q^{1+2+\dots+(n-1)} = a^n q^{\frac{n(n-1)}{2}} = (a^2 q^{n-1})^{\frac{n}{2}} = \left(\frac{S}{R}\right)^{\frac{n}{2}}$$

Economics Entrance Exam - 1988

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These problems were given at the written mathematics entrance examination of the universities and colleges of economics in Hungary in 1988. Problems 1–5 were the same for all applicants; for problems 6–8, there was one version recommended for students from academic high schools and another one for students from vocational secondary schools. The points for each problem are shown in parentheses.

For all applicants

1. Solve the given equation over the real numbers.

$$\sqrt{1 + x\sqrt{x^2 - 24}} = x - 1$$

(9 points)

Solution: We need $x \geq 1$ (the right-hand side is nonnegative) and $x^2 \geq 24$, so $x \geq \sqrt{24}$. Squaring,

$$1 + x\sqrt{x^2 - 24} = x^2 - 2x + 1, \quad x\sqrt{x^2 - 24} = x(x - 2), \quad \sqrt{x^2 - 24} = x - 2$$

(we divided by $x > 0$). Squaring again, $x^2 - 24 = x^2 - 4x + 4$, so $x = 7$. Check: $\sqrt{1 + 7\sqrt{25}} = \sqrt{36} = 6 = 7 - 1$. The solution is $x = 7$.

2. The sum of the first three terms of an arithmetic sequence is 105. If we leave the first two numbers unchanged and add 180 to the third number, then we get the first three terms of a geometric sequence. Find the first three terms of the arithmetic sequence. (11 points)

Solution: Let the terms be $35 - d$, 35 , $35 + d$ (their sum is 105). Then $35 - d$, 35 , $215 + d$ is a geometric sequence, so the square of the middle term is the product of the other two:

$$(35 - d)(215 + d) = 1225, \quad d^2 + 180d - 6300 = 0, \quad (d - 30)(d + 210) = 0$$

If $d = 30$, the arithmetic sequence is 5, 35, 65 (and 5, 35, 245 is geometric with ratio 7). If $d = -210$, the arithmetic sequence is 245, 35, -175 (and 245, 35, 5 is geometric with ratio $\frac{1}{7}$).

Note: In KoMaL, this problem says to add 108 to the third number. That is most likely a misprint for 180: with 108, the equation becomes $d^2 + 108d - 3780 = 0$, whose solutions $d = -54 \pm 6\sqrt{186}$ are not nice numbers.

3. Which real numbers satisfy the given equation?

$$\log(4^{x-2} + 9) - \log(2^{x-2} + 1) + \log 2 - 1 = 0$$

(12 points)

Solution: Both arguments of the logarithm are positive for every x . Since $1 = \log 10$, the equation says

$$\log(2(4^{x-2} + 9)) = \log(10(2^{x-2} + 1)), \quad 2(4^{x-2} + 9) = 10(2^{x-2} + 1)$$

Let $u = 2^{x-2} > 0$; then $4^{x-2} = u^2$, and

$$2u^2 + 18 = 10u + 10, \quad u^2 - 5u + 4 = 0, \quad (u - 1)(u - 4) = 0$$

$u = 1$ gives $x - 2 = 0$, $x = 2$; $u = 4$ gives $x - 2 = 2$, $x = 4$. The solutions are $x = 2$ and $x = 4$.

4. One angle of a triangle is equal to the sum of the other two angles. The radius of the circle circumscribed around the triangle is 6 units, and the area of the square drawn on the shortest side is equal to the area of the triangle. Find the area of the triangle. (12 points)

Solution: If $\gamma = \alpha + \beta$, then $2\gamma = 180^\circ$, so the triangle has a right angle. The hypotenuse is a diameter of the circumscribed circle, so it is 12 units long. Let the legs be $a \leq b$; the shortest side is a . Then

$$a^2 = \frac{ab}{2}, \quad b = 2a$$

and by the Pythagorean theorem $a^2 + 4a^2 = 144$, so $a^2 = \frac{144}{5}$. The area of the triangle is

$$a^2 = \frac{144}{5} = 28.8 \text{ square units}$$

5. In triangle ABC , $AB = BC$. Let F be the midpoint of BC , and let T be the foot of the altitude to the side BC . Find the coordinates of the vertices B and C , if $A(-2, 1)$, $T(16, 1)$, and the second coordinate of F is 13. (14 points)

Solution: The altitude AT is horizontal, so the line BC is vertical: $x = 16$. Let $B = (16, b)$; since $F = (16, 13)$ is the midpoint of BC , $C = (16, 26 - b)$. Then $BC = |2b - 26|$, and $AB = BC$ gives

$$18^2 + (b - 1)^2 = (2b - 26)^2, \quad 3b^2 - 102b + 351 = 0, \quad b^2 - 34b + 117 = 0$$

so $b = 17 \pm \sqrt{172} = 17 \pm 2\sqrt{43}$. There are two triangles:

$$\begin{aligned} B(16, 17 + 2\sqrt{43}) &\approx (16, 30.11), & C(16, 9 - 2\sqrt{43}) &\approx (16, -4.11) \\ B(16, 17 - 2\sqrt{43}) &\approx (16, 3.89), & C(16, 9 + 2\sqrt{43}) &\approx (16, 22.11) \end{aligned}$$

In the first one, T is on the side BC ; in the second one, the angle at B is obtuse and T is on the extension of BC . (In both, $AB = BC = 4\sqrt{43} \pm 8$.)

Recommended for students from academic high schools

6. Let x, y, z be real numbers such that $xyz = 1$ and $1 + z + zx \neq 0$. Prove that the expression

$$\frac{1}{1 + x + xy} + \frac{1}{1 + y + yz} + \frac{1}{1 + z + zx}$$

is defined, and its value does not depend on the choice of x, y, z .

(12 points)

Solution: Since $xyz = 1$, none of x, y, z is zero. Using $xyz = 1$,

$$z(1 + x + xy) = z + zx + xyz = 1 + z + zx, \quad zx(1 + y + yz) = zx + xyz + xyz \cdot z = 1 + z + zx$$

So $1 + x + xy = \frac{1 + z + zx}{z} \neq 0$ and $1 + y + yz = \frac{1 + z + zx}{zx} \neq 0$, and the expression is defined. Its value is

$$\frac{z}{1 + z + zx} + \frac{zx}{1 + z + zx} + \frac{1}{1 + z + zx} = \frac{1 + z + zx}{1 + z + zx} = 1$$

7. The tangents of the angles of a triangle are in the ratio $1 : 2 : 3$. What is the ratio of the sides of the triangle? (15 points)

Solution: Let $\tan \alpha = t$, $\tan \beta = 2t$, $\tan \gamma = 3t$. Since $\gamma = 180^\circ - (\alpha + \beta)$,

$$\tan \gamma = -\tan(\alpha + \beta) = -\frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta}, \quad 3t = -\frac{3t}{1 - 2t^2}$$

$t \neq 0$, so $1 - 2t^2 = -1$, $t^2 = 1$. If $t = -1$, all three tangents would be negative, that is, all three angles would be obtuse, which is impossible. So $\tan \alpha = 1$, $\tan \beta = 2$, $\tan \gamma = 3$, all three angles are acute, and

$$\sin \alpha = \frac{1}{\sqrt{2}}, \quad \sin \beta = \frac{2}{\sqrt{5}}, \quad \sin \gamma = \frac{3}{\sqrt{10}}$$

(using $\sin \varphi = \frac{\tan \varphi}{\sqrt{1 + \tan^2 \varphi}}$ for acute angles). By the Law of Sines, the sides are in the ratio of the sines of the opposite angles; multiplying by $\sqrt{10}$:

$$a : b : c = \frac{1}{\sqrt{2}} : \frac{2}{\sqrt{5}} : \frac{3}{\sqrt{10}} = \sqrt{5} : 2\sqrt{2} : 3$$

8. Let P be an arbitrary point in the plane of the quadrilateral $ABCD$. The reflection of P through A is Q , and the reflection of P through B is R . Prove that the quadrilateral $ABCD$ is a parallelogram if and only if the reflection of Q through D is the same point as the reflection of R through C . (15 points)

Solution: Use position vectors (vectors from a fixed origin), denoted by the lowercase letters. The reflection of a point X through a point M is $2\mathbf{m} - \mathbf{x}$ (because M is the midpoint). So

$$\mathbf{q} = 2\mathbf{a} - \mathbf{p}, \quad \mathbf{r} = 2\mathbf{b} - \mathbf{p}$$

The reflection of Q through D is $2\mathbf{d} - \mathbf{q} = 2\mathbf{d} - 2\mathbf{a} + \mathbf{p}$, and the reflection of R through C is $2\mathbf{c} - \mathbf{r} = 2\mathbf{c} - 2\mathbf{b} + \mathbf{p}$. These two points are the same if and only if

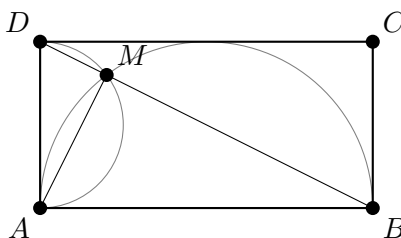
$$\mathbf{d} - \mathbf{a} = \mathbf{c} - \mathbf{b}, \quad \text{that is, } \overrightarrow{AD} = \overrightarrow{BC}$$

and the quadrilateral $ABCD$ is a parallelogram if and only if $\overrightarrow{AD} = \overrightarrow{BC}$ (the sides AD and BC are parallel, equal in length, and point in the same direction). (So the statement is true for every choice of P .)

Recommended for students from vocational secondary schools

6. In rectangle $ABCD$, $AB = 2a$ and $BC = a$. Circles are drawn on the sides AB and AD as diameters; their intersection point inside the rectangle is M . Show that M is on the diagonal BD . How long are the segments MA , MB , and MD ? (13 points)

Solution: By Thales' theorem, $\angle AMB = 90^\circ$ (since M is on the circle with diameter AB) and $\angle AMD = 90^\circ$ (since M is on the circle with diameter AD). So both MB and MD are perpendicular to MA at M , so B, M, D are on one line: M is on the diagonal BD , and it is the foot of the perpendicular from A to BD .



$BD = \sqrt{4a^2 + a^2} = \sqrt{5}a$. In the right triangle ABD , the altitude is AM , so (computing the area in two ways, and using the similar triangles)

$$MA = \frac{AB \cdot AD}{BD} = \frac{2a^2}{\sqrt{5}a} = \frac{2\sqrt{5}}{5}a, \quad MB = \frac{AB^2}{BD} = \frac{4\sqrt{5}}{5}a, \quad MD = \frac{AD^2}{BD} = \frac{\sqrt{5}}{5}a$$

7. Solve the given equation over the real numbers.

$$x^2 + 2x \sin xy + 1 = 0$$

(14 points)

Solution: Since $1 = \sin^2 xy + \cos^2 xy$, the equation can be written as

$$(x + \sin xy)^2 + \cos^2 xy = 0$$

A sum of two squares is zero only if both are zero: $\cos xy = 0$ and $x = -\sin xy$. Then $\sin xy = \pm 1$, so $x = \pm 1$.

If $x = 1$: $\sin y = -1$. If $x = -1$: $\sin(-y) = 1$, that is, $\sin y = -1$ again. The solutions are

$$x = 1 \text{ or } x = -1, \quad y = -\frac{\pi}{2} + 2k\pi, \quad k \text{ is an integer}$$

8. Prove that if the sum of the real numbers a and b is 1, then

$$a^4 + b^4 \geq \frac{1}{8}$$

(15 points)

Solution: For any real numbers u and v , $u^2 + v^2 \geq \frac{(u+v)^2}{2}$, because this is equivalent to $(u-v)^2 \geq 0$. Using this twice,

$$a^4 + b^4 \geq \frac{(a^2 + b^2)^2}{2} \geq \frac{1}{2} \left(\frac{(a+b)^2}{2} \right)^2 = \frac{1}{2} \cdot \frac{1}{4} = \frac{1}{8}$$

Equality holds for $a = b = \frac{1}{2}$.