

Written Final and Entrance Exam - 1990 (I)

translated from: KoMaL, December 1990, pages 442-443

Pdf file: KoMaL, December 1990, pages 442-443

These problems were given at the joint written final (érettségi) and entrance examination in mathematics in Hungary in 1990. KoMaL published them in December 1990 as practice set IV ("Matematika felvételre előkészítő feladatok - 1990. - IV.") of its series for students preparing for the entrance examination.

1. Solve the given equation over the real numbers.

$$x^2 + 3x + |x + 3| = 0$$

Solution: If $x \geq -3$: $x^2 + 4x + 3 = (x + 1)(x + 3) = 0$, so $x = -1$ or $x = -3$, and both are allowed. If $x < -3$: $x^2 + 2x - 3 = (x + 3)(x - 1) = 0$, so $x = -3$ or $x = 1$, but neither is less than -3 . The solutions are $x = -3$ and $x = -1$.

2. The price of a product was first raised by 10%, then lowered by 10%. Finally it was raised by 10% again. What percent of the original price is the final price?

Solution: If the original price is x , the final price is $x \cdot 1.1 \cdot 0.9 \cdot 1.1 = 1.089x$, so the final price is 108.9% of the original price.

3. Find the area of triangle ABC in which $AB = 3$, $BC = 7$, and the median from vertex B is 4 units long.

Solution: Let F be the midpoint of AC , and reflect B through F to get B' . Then $ABCB'$ is a parallelogram (its diagonals bisect each other), so $CB' = AB = 3$, $BB' = 2BF = 8$, and the triangle BCB' has the same area as ABC (each is half of the parallelogram). The sides of triangle BCB' are 7, 3, 8, so by Heron's formula with $s = 9$,

$$T = \sqrt{9 \cdot 2 \cdot 6 \cdot 1} = \sqrt{108} = 6\sqrt{3} \approx 10.39$$

4. Solve the following system of equations over the pairs of real numbers.

$$\begin{aligned} x^{-1} + y^{-1} &= x + y \\ (2 + \log y) \log x &= 1 \end{aligned}$$

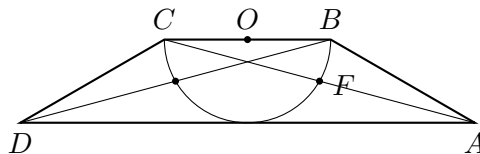
Solution: From the second equation, $x > 0$ and $y > 0$. The first equation is $\frac{x + y}{xy} = x + y$, and since $x + y > 0$, $xy = 1$, so $\log y = -\log x$. Then the second equation becomes

$$(2 - \log x) \log x = 1, \quad (\log x - 1)^2 = 0$$

so $\log x = 1$, and the only solution is $x = 10$, $y = \frac{1}{10}$.

5. The parallel sides of trapezoid $ABCD$ are AD and BC . A circle is drawn on the side BC of the trapezoid as diameter. This circle is tangent to the side AD and bisects both diagonals. Find the angles of the trapezoid.

Solution: Let O be the midpoint of BC (the center of the circle), $r = \frac{BC}{2}$ the radius, and F the midpoint of the diagonal AC . F is on the circle, so by Thales' theorem $\angle BFC = 90^\circ$: BF is perpendicular to AC and goes through its midpoint, so $BA = BC$. Similarly, the midpoint of BD is on the circle, so $CD = CB$. The circle is tangent to AD , so the height of the trapezoid is $r = \frac{BC}{2}$. So each leg is twice the height, and makes a 30° angle with the base AD :



The angles of the trapezoid are 30° at A and D , and 150° at B and C . (The legs could also lean in the same direction, but then $AD = BC$ and the figure is a parallelogram with angles 30° and 150° .)

6. Find the equation of the line that goes through the point $(6, 4)$ and intersects the lines $x + y = 4$ and $x + y = 5$ at points whose first coordinates differ by 2.

Solution: The vertical line $x = 6$ does not work (the first coordinates would be equal), and a line with slope -1 is parallel to the given lines. So the line is $y = m(x - 6) + 4$ with $m \neq -1$. It intersects $x + y = 4$ at $x_1 = \frac{6m}{m+1}$ and $x + y = 5$ at $x_2 = \frac{6m+1}{m+1}$. So

$$|x_2 - x_1| = \frac{1}{|m+1|} = 2, \quad m = -\frac{1}{2} \text{ or } m = -\frac{3}{2}$$

There are two such lines:

$$x + 2y = 14 \quad \text{and} \quad 3x + 2y = 26$$

7. Solve the following equation over the real numbers.

$$\log_5(x-4) + \log_{\sqrt{5}}(x^3-2) + \log_{0.2}(x-4) = 4$$

Solution: We need $x > 4$. Since $\log_{0.2}(x-4) = -\log_5(x-4)$, the equation becomes $\log_{\sqrt{5}}(x^3-2) = 4$, that is, $x^3 - 2 = (\sqrt{5})^4 = 25$, so $x = 3$. But $x = 3$ does not satisfy $x > 4$, so the equation has no real solution.

8. Find all pairs of real numbers (a, b) for which the equation

$$\cos(ax + b^2) - (a \cdot \cos x + b^2) = 1 - a$$

holds for every real number x .

Solution: For $x = 0$: $\cos b^2 - a - b^2 = 1 - a$, that is, $\cos b^2 = 1 + b^2$. Since $\cos b^2 \leq 1 \leq 1 + b^2$, this means $b = 0$. Then the condition is $\cos ax - a \cos x = 1 - a$, that is, $1 - \cos ax = a(1 - \cos x)$, or

$$\sin^2 \frac{ax}{2} = a \sin^2 \frac{x}{2} \quad \text{for every } x$$

For $x = 2\pi$: $\sin^2 a\pi = 0$, so a is an integer. For $x = \pi$: $\sin^2 \frac{a\pi}{2} = a$, and the left-hand side is 0 or 1 for an integer a , so $a = 0$ or $a = 1$. Both work: for $a = 0$ the equation is $1 = 1$, and for $a = 1$ it is $0 = 0$. The pairs are $(0, 0)$ and $(1, 0)$.

Written Final and Entrance Exam - 1990 (II)

translated from: KoMaL, January 1991, pages 7-8

Pdf file: KoMaL, January 1991, pages 7-8

These problems were also given at the joint written final (érettségi) and entrance examination in mathematics in Hungary in 1990 (this is a second problem set from that year). KoMaL published them in January 1991 as practice set V ("Matematikai felvételtelre előkészítő feladatok - 1991. - V."). The points for each problem are shown in parentheses.

1. The legs of a right triangle are 3 and $3\sqrt{3}$ units long. Find the perimeter and the area of the circular segment that the shorter leg of the triangle cuts off from the circle circumscribed around the triangle. (The circular segment is the one that is smaller than a half disk.) (10 points)

Solution: The hypotenuse is $\sqrt{9+27} = 6$, and it is a diameter of the circumscribed circle, so the radius is 3. The shorter leg is a chord of length 3, equal to the radius, so with the center O it forms an equilateral triangle, and the central angle over it is 60° . The perimeter of the segment is the chord plus the arc:

$$K = 3 + \frac{2 \cdot 3\pi}{6} = 3 + \pi \approx 6.14$$

The area is the area of the sector minus the area of the equilateral triangle:

$$T = \frac{9\pi}{6} - \frac{9\sqrt{3}}{4} = \frac{6\pi - 9\sqrt{3}}{4} \approx 0.815$$

2. Solve the given equation over the real numbers.

$$\cot x - \sin 2x = \cot x \cdot \cos 2x$$

(10 points)

Solution: We need $\sin x \neq 0$, that is, $x \neq k\pi$. The equation is $\cot x (1 - \cos 2x) = \sin 2x$, and

$$\cot x (1 - \cos 2x) = \frac{\cos x}{\sin x} \cdot 2 \sin^2 x = 2 \sin x \cos x = \sin 2x$$

So the equation is true for every allowed x : the solution is every real number $x \neq k\pi$ (k is an integer).

3. Four siblings are asked about their ages. The oldest one says: "Our birth years are the first four terms of an arithmetic sequence." The next one in age says: "15 years ago, the ages of my siblings were equal to the first three terms of a geometric sequence, and now the sum of their ages is $\frac{11}{4}$ times my age." Find the present ages of the siblings. (12 points)

Solution: The present ages also form an arithmetic sequence: $a, a + d, a + 2d, a + 3d$, where a is the age of the oldest. The second oldest is $a + d$, and the other three siblings are $a, a + 2d, a + 3d$. The conditions are

$$(a + 2d - 15)^2 = (a - 15)(a + 3d - 15), \quad 3a + 5d = \frac{11}{4}(a + d)$$

With $u = a - 15$, the first one is $(u + 2d)^2 = u(u + 3d)$, that is, $d(u + 4d) = 0$. The second one gives $a + 9d = 0$. If $d = 0$, then $a = 0$, which is impossible. So $a - 15 + 4d = 0$ and $a = -9d$, which give $d = -3, a = 27$. The siblings are now 27, 24, 21, and 18 years old. (Check: 15 years ago, the oldest, third and fourth siblings were 12, 6, 3; now $27 + 21 + 18 = 66 = \frac{11}{4} \cdot 24$.)

4. The side AB of a triangle is 10, the median to it is 6, and another median is 9 units long. Find the sides AC and BC of the triangle. (12 points)

Solution: Let $a = BC$, $b = CA$, $c = AB = 10$. The length of the median to the side c is given by $m_c^2 = \frac{2a^2 + 2b^2 - c^2}{4}$ (this follows from the Law of Cosines, or from the parallelogram law). So

$$\frac{2a^2 + 2b^2 - 100}{4} = 36, \quad a^2 + b^2 = 122$$

If the other median is the one from A : $m_a^2 = \frac{2b^2 + 2c^2 - a^2}{4} = 81$, so $2b^2 - a^2 = 124$. Then $3b^2 = 246$, $b^2 = 82$, $a^2 = 40$:

$$AC = \sqrt{82} \approx 9.06, \quad BC = \sqrt{40} = 2\sqrt{10} \approx 6.32$$

If the other median is the one from B , the roles of A and B are exchanged: $AC = 2\sqrt{10}$ and $BC = \sqrt{82}$.

5. The points $A(-4, -1)$ and $B(6, 1)$ are on the circle $x^2 + y^2 - 2x - 25 = 0$. The chords AC and BC of the circle are in the ratio 3 : 2. Find the coordinates of the point C . (13 points)

Solution: Let $C(x, y)$ be on the circle, so $x^2 + y^2 = 2x + 25$. The condition $2AC = 3BC$ means

$$4[(x+4)^2 + (y+1)^2] = 9[(x-6)^2 + (y-1)^2]$$

Using $x^2 + y^2 = 2x + 25$ on both sides, this becomes $4(10x + 2y + 42) = 9(62 - 10x - 2y)$, that is, $5x + y = 15$. Substituting $y = 15 - 5x$ into the equation of the circle:

$$13x^2 - 76x + 100 = 0, \quad x = 2 \text{ or } x = \frac{50}{13}$$

So $C_1(2, 5)$ or $C_2\left(\frac{50}{13}, -\frac{55}{13}\right)$.

6. A frustum of a cone is circumscribed around a sphere so that the base circle and the top circle of the frustum each touch the sphere at one point, and its lateral surface touches the sphere along a circle. What is the quotient of the volume and the surface area of the frustum, and what is the product of the radii of the base and top circles, if the radius of the sphere is 6 units? (14 points)

Solution: Let R and r be the radii of the base and the top. The height of the frustum is the diameter of the sphere, 12. In the cross-section through the axis, the slant height is a tangent to the circle, and the tangent segments from its endpoints are R and r , so the slant height is $R + r$. The right triangle cut off by the height from the end of the top circle has legs 12 and $R - r$ and hypotenuse $R + r$:

$$144 + (R - r)^2 = (R + r)^2, \quad Rr = 36$$

The volume and the surface area are

$$V = \frac{12\pi}{3} (R^2 + Rr + r^2), \quad A = \pi (R^2 + r^2) + \pi (R + r) (R + r) = 2\pi (R^2 + Rr + r^2)$$

so $\frac{V}{A} = 2$, and $Rr = 36$. (In general, for a solid circumscribed around a sphere of radius ρ , $V = \frac{\rho A}{3}$.)

7. For what values of the real parameter p does the equation

$$1 + \frac{x}{x-p} = \frac{2p}{x+p} + \frac{2x^2 - 4x + 9p^2}{x^2 - p^2}$$

have a solution? For which values of p is the root of the equation less than p ?

(14 points)

Solution: We need $x \neq \pm p$. Multiplying by $x^2 - p^2$:

$$x^2 - p^2 + x(x+p) = 2p(x-p) + 2x^2 - 4x + 9p^2, \quad x(4-p) = 8p^2$$

If $p = 4$, there is no solution. Otherwise $x = \frac{8p^2}{4-p}$, and this is a solution unless it equals p or $-p$. $\frac{8p^2}{4-p} = p$ means $9p^2 = 4p$, that is, $p = 0$ or $p = \frac{4}{9}$; $\frac{8p^2}{4-p} = -p$ means $7p^2 = -4p$, that is, $p = 0$ or $p = -\frac{4}{7}$. So the equation has a solution exactly when $p \neq -\frac{4}{7}, 0, \frac{4}{9}, 4$.

The root is less than p if $\frac{8p^2}{4-p} < p$. For $p < 4$ this means $8p^2 < 4p - p^2$, that is, $0 < p < \frac{4}{9}$; for $p > 4$ it means $8p^2 > 4p - p^2$, which is true. So the root is less than p for $0 < p < \frac{4}{9}$ and for $p > 4$.

8. Prove that if n is a positive integer, then

a) 81 is a divisor of $10^n(9n-1)+1$,

b) 1990 is a divisor of $(600^n - 3^n)(n^5 - n)$.

(15 points)

Solution: a) $10^n(9n-1)+1 = 9n \cdot 10^n - (10^n - 1)$, and $10^n - 1 = 9(10^{n-1} + \dots + 10 + 1)$, so

$$10^n(9n-1)+1 = 9[(10^n - 10^{n-1}) + (10^n - 10^{n-2}) + \dots + (10^n - 1)]$$

(the bracket has n terms). Each difference $10^n - 10^k = 10^k(10^{n-k} - 1)$ is divisible by 9, so the expression is divisible by $9 \cdot 9 = 81$.

b) $1990 = 10 \cdot 199$, and 10 and 199 are relatively prime. $600^n - 3^n$ is divisible by $600 - 3 = 597 = 3 \cdot 199$ (since $a^n - b^n$ is divisible by $a - b$), so it is divisible by 199. And $n^5 - n$ is divisible by 10: n^5 always has the same last digit as n (check the ten possible last digits: $0^5, 1^5, 2^5 = 32, \dots, 9^5 = 59049$). So the product is divisible by $10 \cdot 199 = 1990$.

Written Final and Entrance Exam - 1990 (III)

translated from: KoMaL, February 1991, pages 63-64

Pdf file: KoMaL, February 1991, pages 63-64

These problems were also given at the joint written final (érettségi) and entrance examination in mathematics in Hungary in 1990 (this is a third problem set from that year). KoMaL published them in February 1991 as practice set VI ("Matematikai felvételtire előkészítő feladatok - 1991. - VI.").

1. Solve the following system of equations over the real numbers.

$$\frac{3}{\sqrt{x-2}} - \frac{5}{\sqrt{y+1}} = -\frac{1}{6}, \quad \frac{4}{\sqrt{x-2}} + \frac{2}{\sqrt{y+1}} = \frac{8}{3}$$

Solution: Let $u = \frac{1}{\sqrt{x-2}} > 0$ and $v = \frac{1}{\sqrt{y+1}} > 0$. Then $3u - 5v = -\frac{1}{6}$ and $4u + 2v = \frac{8}{3}$. From the second equation $v = \frac{4}{3} - 2u$, and then $13u - \frac{20}{3} = -\frac{1}{6}$, so $u = \frac{1}{2}$ and $v = \frac{1}{3}$. So $\sqrt{x-2} = 2$ and $\sqrt{y+1} = 3$:

$$x = 6, \quad y = 8$$

2. Solve the following equation over the real numbers.

$$729 \cdot 3^x - 2^{2x+12} = 0$$

Solution: $729 = 3^6$ and $2^{2x+12} = 4^{x+6}$, so the equation is $3^{x+6} = 4^{x+6}$, that is, $\left(\frac{3}{4}\right)^{x+6} = 1$. So $x + 6 = 0$, and the solution is $x = -6$.

3. For which real numbers x are the following expressions defined?

$$\text{a) } \frac{\sqrt{-x^2 + 8x - 16}}{x + 4} \quad \text{b) } \log \left| \frac{1 - 3^x}{3^x} \right| \quad \text{c) } \frac{1}{\sqrt{\log(1 - \cos^2 x)}}$$

Solution: a) $-x^2 + 8x - 16 = -(x - 4)^2 \geq 0$ only for $x = 4$, and then $x + 4 \neq 0$. The expression is defined only for $x = 4$.

b) We need $\frac{1 - 3^x}{3^x} \neq 0$, that is, $3^x \neq 1$. The expression is defined for every $x \neq 0$.

c) $1 - \cos^2 x = \sin^2 x \leq 1$, so $\log(1 - \cos^2 x) \leq 0$ wherever it is defined, but we would need it to be positive. The expression is not defined for any x .

4. We deposit one hundred thousand forints in a bank, at 16% annual interest. After four years, we withdraw 120 thousand forints. After that, how long do we have to wait until our deposit is more than one hundred thousand forints again?

Solution: After four years the deposit is $100\,000 \cdot 1.16^4 \approx 181\,064$ forints, so after the withdrawal it is about 61 064 forints. After t more years it is $61\,064 \cdot 1.16^t$, and we need

$$61\,064 \cdot 1.16^t > 100\,000, \quad t > \frac{\log \frac{100\,000}{61\,064}}{\log 1.16} \approx 3.32$$

Since the interest is added at the end of each year (after 3 years the deposit is only about 95 314 forints, and after 4 years about 110 565 forints), we have to wait 4 years.

5. Solve the equation

$$(1 + \sin 2x) \cdot (\cos x - \sin x) = \cos x + \sin x$$

over the real numbers.

Solution: $1 + \sin 2x = \sin^2 x + \cos^2 x + 2 \sin x \cos x = (\cos x + \sin x)^2$, so the equation is

$$(\cos x + \sin x) [(\cos x + \sin x)(\cos x - \sin x) - 1] = 0, \quad (\cos x + \sin x)(\cos 2x - 1) = 0$$

$\cos x + \sin x = 0$ means $\tan x = -1$, that is, $x = \frac{3\pi}{4} + k\pi$; $\cos 2x = 1$ means $x = l\pi$. The solutions are

$$x = \frac{3\pi}{4} + k\pi \quad \text{or} \quad x = l\pi, \quad k, l \text{ are integers}$$

6. Two adjacent vertices of rectangle $ABCD$ are $A\left(\frac{3}{2}, 1\right)$ and $B\left(\frac{9}{2}, 0\right)$. The ratio of the lengths of the sides AB and BC is 1 : 3. Find the coordinates of the points C and D .

Solution: $\overrightarrow{AB} = (3, -1)$. The vector $\overrightarrow{BC} = \overrightarrow{AD}$ is perpendicular to it and three times as long, so it is $3(1, 3) = (3, 9)$ or $(-3, -9)$. There are two such rectangles:

$$\begin{aligned} \overrightarrow{BC} = (3, 9) & : C\left(\frac{15}{2}, 9\right), D\left(\frac{9}{2}, 10\right) \\ \overrightarrow{BC} = (-3, -9) & : C\left(\frac{3}{2}, -9\right), D\left(-\frac{3}{2}, -8\right) \end{aligned}$$

7. For what values of the parameter k does the quadratic equation

$$x^2 - 2(3k + 5)x + k^2 + 13k - 90 = 0$$

have two real roots with opposite signs?

Solution: The roots are real for every k , since $\frac{D}{4} = (3k + 5)^2 - (k^2 + 13k - 90) = 8k^2 + 17k + 115$ has a negative discriminant ($17^2 - 32 \cdot 115 < 0$), so it is always positive. The roots have opposite signs exactly when their product is negative:

$$k^2 + 13k - 90 = (k + 18)(k - 5) < 0, \quad -18 < k < 5$$

(In fact, a negative product already implies two real roots.)

8. Solve the following inequality over the real numbers.

$$\log_{\frac{2}{3}} (\log_{16} (x^2 - 48)) \geq 0$$

Solution: Since $\log_{\frac{2}{3}}$ is decreasing and $\log_{\frac{2}{3}} 1 = 0$, the inequality means $0 < \log_{16} (x^2 - 48) \leq 1$, that is, $1 < x^2 - 48 \leq 16$, $49 < x^2 \leq 64$. The solution is

$$-8 \leq x < -7 \quad \text{or} \quad 7 < x \leq 8$$