

# Economics Entrance Exam - 1991

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These problems were given at the written mathematics entrance examination of the universities and colleges of economics in Hungary in 1991. The points for each problem are shown in parentheses.

1. A factory makes a part in two quality classes. 78% of the revenue comes from the first class product. By what percent does the revenue of the factory increase if the production of the first class product is increased by 27% and the production of the second class product by 22%? (8 points)

Solution: (We assume that the prices do not change.) If the revenue was 100 units, 78 came from the first class and 22 from the second class product. The new revenue is

$$78 \cdot 1.27 + 22 \cdot 1.22 = 99.06 + 26.84 = 125.9$$

so the revenue increases by 25.9%.

2. Find the exact values of the following expressions (without using tables or a calculator).

a)  $\log_3 \left( -\tan \frac{1991\pi}{4} \right)$  (3 points)

b)  $1991\sqrt{\cos^2 9\pi - 1}$  (3 points)

c)  $\log_{2+\sqrt{3}} (2 - \sqrt{3})$  (5 points)

Solution: a)  $\frac{1991\pi}{4} = 497\pi + \frac{3\pi}{4}$ , and the period of  $\tan$  is  $\pi$ , so  $\tan \frac{1991\pi}{4} = \tan \frac{3\pi}{4} = -1$ . The value is  $\log_3 1 = 0$ .

b)  $\cos 9\pi = -1$ , so  $\cos^2 9\pi - 1 = 0$ , and the value is 0.

c)  $(2 + \sqrt{3})(2 - \sqrt{3}) = 4 - 3 = 1$ , so  $2 - \sqrt{3} = (2 + \sqrt{3})^{-1}$ , and the value is  $-1$ .

3. The radii  $CA$  and  $CB$  of a circle with center  $C$  and radius 3 units make a  $120^\circ$  angle with each other. A cone is placed on the plane of the circle so that its base circle is tangent to the arc  $AB$  and to the segments  $CA$  and  $CB$ . The height of the cone is equal to the length of  $AB$ . Find the volume of the cone. (12 points)

Solution: The base circle of the cone is inscribed in the circular sector  $ACB$ . Its center  $K$  is on the bisector of the angle  $ACB$ , at distance  $d$  from  $C$ , and its radius is  $\rho = d \sin 60^\circ = \frac{\sqrt{3}}{2}d$  (the distance of  $K$  from the radius  $CA$ ). It touches the arc, so  $d + \rho = 3$ . So

$$\rho = \frac{3 \cdot \frac{\sqrt{3}}{2}}{1 + \frac{\sqrt{3}}{2}} = \frac{3\sqrt{3}}{2 + \sqrt{3}} = 3\sqrt{3}(2 - \sqrt{3}) = 6\sqrt{3} - 9 \approx 1.392$$

The height is the chord  $AB = 2 \cdot 3 \sin 60^\circ = 3\sqrt{3}$ . The volume of the cone is

$$V = \frac{1}{3}\rho^2\pi \cdot 3\sqrt{3} = \sqrt{3}\pi (6\sqrt{3} - 9)^2 = \sqrt{3}\pi (189 - 108\sqrt{3}) = (189\sqrt{3} - 324)\pi \approx 10.55$$

4. Solve the given equation.

$$8 \sin^2 \frac{x}{2} - 2 \cos x = 7$$

(13 points)

Solution: Since  $2 \sin^2 \frac{x}{2} = 1 - \cos x$ , the equation is  $4(1 - \cos x) - 2 \cos x = 7$ , that is,  $\cos x = -\frac{1}{2}$ . The solutions are

$$x = \pm \frac{2\pi}{3} + 2k\pi, \quad k \text{ is an integer}$$

5. Let  $a_1, a_2, a_3, \dots, a_n, \dots$  be an arithmetic sequence. Prove that the sequence  $b_1, b_2, b_3, \dots, b_n, \dots$  defined by  $b_n = a_{n+1}^2 - a_n^2$  is also an arithmetic sequence. (13 points)

Solution: Let  $d$  be the difference of the sequence  $(a_n)$ . Then

$$b_n = (a_{n+1} - a_n)(a_{n+1} + a_n) = d(a_{n+1} + a_n)$$

so

$$b_{n+1} - b_n = d(a_{n+2} + a_{n+1} - a_{n+1} - a_n) = d(a_{n+2} - a_n) = d \cdot 2d = 2d^2$$

which does not depend on  $n$ . So  $(b_n)$  is an arithmetic sequence with difference  $2d^2$ .

6. The axis of symmetry of the kite  $ABCD$  is the diagonal  $AC$ , where  $A(0, 0)$  and  $C(8, 10)$ . The area of the kite is 41 square units. One diagonal divides the other in the ratio 3 : 2, counting from the origin. Find the coordinates of the missing vertices. (13 points)

Solution: The axis of symmetry  $AC$  bisects  $BD$ , so it is  $BD$  that divides  $AC$  in the ratio 3 : 2 from  $A$ : the intersection point of the diagonals is  $M = \frac{3}{5}C = \left(\frac{24}{5}, 6\right)$ .  $AC = \sqrt{64 + 100} = 2\sqrt{41}$ , and the diagonals are perpendicular, so the area is  $\frac{AC \cdot BD}{2} = 41$ , which gives  $BD = \sqrt{41}$ . The direction of  $BD$  is perpendicular to  $(8, 10)$ , that is, parallel to  $(5, -4)$ , whose length is  $\sqrt{41}$ . So  $B$  and  $D$  are  $M \pm \frac{1}{2}(5, -4)$ :

$$\left(\frac{73}{10}, 4\right) \quad \text{and} \quad \left(\frac{23}{10}, 8\right)$$

7. Which triples of real numbers satisfy the following system of equations?

$$\begin{aligned} x + y + z &= 2 \\ 2xy &= z^2 + 4 \end{aligned}$$

(14 points)

Solution:  $x + y = 2 - z$  and  $xy = \frac{z^2 + 4}{2}$ . For real  $x$  and  $y$ ,  $(x + y)^2 \geq 4xy$  (since  $(x - y)^2 \geq 0$ ), so

$$(2 - z)^2 \geq 2(z^2 + 4), \quad 0 \geq z^2 + 4z + 4 = (z + 2)^2$$

So  $z = -2$ , and then  $x + y = 4$ ,  $xy = 4$ ,  $x = y = 2$ . The only solution is  $(x, y, z) = (2, 2, -2)$ .

8. Solve the following equation over the real numbers.

$$\log_{1991}(x-3) + \log_{1992}(x-3) = 3 - \log(x^5 - 24)$$

(16 points)

Solution: We need  $x > 3$  (then  $x^5 - 24 > 0$  too). The left-hand side is increasing in  $x$ , and the right-hand side is decreasing, so there is at most one solution. For  $x = 4$ : the left-hand side is  $0 + 0 = 0$ , and the right-hand side is  $3 - \log 1000 = 0$ . So the only solution is  $x = 4$ .

Note: In KoMaL, the last term is printed as  $\log(x^2 - 24)$ . That is most likely a misprint for  $x^5$  (note that  $4^5 - 24 = 1000$ ). With  $x^2$ , the same argument shows that there is exactly one solution, but it is not a nice number:  $x \approx 15.50$ .

Technical Universities Entrance Exam - 1991

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These problems were given at the written mathematics entrance examination of the technical universities and colleges in Hungary in 1991. The points for each problem are shown in parentheses.

1. Solve the following equation over the real numbers.

$$\frac{\sqrt{x+1}+2}{\sqrt{x+1}-1} = \frac{x+1}{x-2}$$

(10 points)

Solution: Let  $t = \sqrt{x+1} \geq 0$ ; then  $x+1 = t^2$  and  $x-2 = t^2-3$ , and we need  $t \neq 1$ ,  $t^2 \neq 3$ . The equation is

$$\frac{t+2}{t-1} = \frac{t^2}{t^2-3}, \quad (t+2)(t^2-3) = t^2(t-1), \quad 3t^2-3t-6=0$$

so  $t = 2$  or  $t = -1$ . Only  $t = 2$  is allowed, which gives  $x = 3$ . (Check:  $\frac{4}{1} = \frac{4}{1}$ .)

2. The third term of an arithmetic sequence is the square of the first term, and the sum of the first three terms is 30. Write down the first three terms of the sequence. (10 points)

Solution: The sum of the first three terms is  $3a_2 = 30$ , so  $a_2 = 10$  and  $a_1 + a_3 = 20$ . With  $a_3 = a_1^2$ :  $a_1^2 + a_1 - 20 = 0$ , so  $a_1 = 4$  or  $a_1 = -5$ . The sequences are

$$4, 10, 16 \quad \text{and} \quad -5, 10, 25$$

3. The base  $BC$  of the isosceles triangle  $ABC$  is 40 units long, and the height to the base is 15 units. The feet of the perpendiculars from the midpoint  $F$  of the base to the legs are  $P$  and  $Q$ . Find the perimeter and the area of the quadrilateral  $APFQ$ . (12 points)

Solution: In the right triangle  $ABF$  (right angle at  $F$ ),  $AF = 15$ ,  $BF = 20$ , so  $AB = 25$ .  $FP$  is the altitude to the hypotenuse, so

$$FP = \frac{AF \cdot BF}{AB} = \frac{15 \cdot 20}{25} = 12, \quad AP = \frac{AF^2}{AB} = \frac{225}{25} = 9$$

By symmetry  $AQ = 9$  and  $FQ = 12$ . The quadrilateral  $APFQ$  consists of the two congruent right triangles  $APF$  and  $AQF$ , so its perimeter is  $2(9+12) = 42$  units, and its area is  $2 \cdot \frac{9 \cdot 12}{2} = 108$  square units.

4. One vertex of a rhombus is  $A(5, 8)$ , and the equation of the line of the diagonal  $BD$  is  $x - 2y + 6 = 0$ . The side of the rhombus is 5 units. Find the coordinates of the other vertices and the area of the rhombus. (13 points)

Solution: The diagonals of a rhombus are perpendicular bisectors of each other. The foot of the perpendicular from  $A$  to the line  $BD$  (whose normal vector is  $(1, -2)$ ) is the center  $M$  of the rhombus:  $M = A + t(1, -2)$  with  $(5 + t) - 2(8 - 2t) + 6 = 0$ , so  $t = 1$  and  $M = (6, 6)$ . Then  $C = 2M - A = (7, 4)$ , and  $AM = \sqrt{5}$ . In the right triangle  $ABM$ ,  $BM = \sqrt{25 - 5} = 2\sqrt{5}$ . The direction of  $BD$  is  $(2, 1)$ , of length  $\sqrt{5}$ , so  $B$  and  $D$  are  $M \pm 2(2, 1)$ :

$$B(10, 8), C(7, 4), D(2, 4)$$

(or  $B$  and  $D$  exchanged). The area is  $\frac{AC \cdot BD}{2} = \frac{2\sqrt{5} \cdot 4\sqrt{5}}{2} = 20$  square units.

5. Solve the following system of equations over the pairs of real numbers.

$$\frac{x}{y} + \frac{y}{x} = \frac{5}{2}; \quad \log_3(x - y) + \log_3(x + y) = 1$$

(13 points)

Solution: We need  $x - y > 0$  and  $x + y > 0$ . With  $t = \frac{x}{y}$ , the first equation is  $t + \frac{1}{t} = \frac{5}{2}$ , so  $t = 2$  or  $t = \frac{1}{2}$ . The second equation says  $(x - y)(x + y) = x^2 - y^2 = 3$ .

If  $x = 2y$ :  $3y^2 = 3$ ,  $y = \pm 1$ .  $(2, 1)$  is good, but  $(-2, -1)$  is not, since  $x + y < 0$ .

If  $y = 2x$ :  $-3x^2 = 3$ , which is impossible.

The only solution is  $x = 2, y = 1$ .

6. All edges of a regular square pyramid are equal. A cube is inscribed in the pyramid so that its base is on the base of the pyramid, and the vertices of its top face are on the lateral edges of the pyramid. How many times the volume of the cube is the volume of the pyramid? (13 points)

Solution: Let the edges of the pyramid be  $a$ . Half of the diagonal of the base is  $\frac{a}{\sqrt{2}}$ , so the height is  $m = \sqrt{a^2 - \frac{a^2}{2}} = \frac{a}{\sqrt{2}}$ . The top face of the cube (with edge  $s$ ) is the cross-section of the pyramid at height  $s$ , which is a square with side  $a\left(1 - \frac{s}{m}\right)$ . So

$$s = a - \sqrt{2}s, \quad s = \frac{a}{1 + \sqrt{2}} = (\sqrt{2} - 1)a$$

The volumes are  $V = \frac{1}{3}a^2 \cdot \frac{a}{\sqrt{2}} = \frac{a^3}{3\sqrt{2}}$  and  $V_{\text{cube}} = (\sqrt{2} - 1)^3 a^3 = (5\sqrt{2} - 7)a^3$ , so

$$\frac{V}{V_{\text{cube}}} = \frac{1}{3\sqrt{2}(5\sqrt{2} - 7)} = \frac{1}{30 - 21\sqrt{2}} = \frac{30 + 21\sqrt{2}}{18} = \frac{10 + 7\sqrt{2}}{6} \approx 3.32$$

7. Prove that for any value of the real parameter  $p$ , the following equation has exactly one real root.

$$x|x + 2p| = p$$

(14 points)

Solution: If  $p = 0$ :  $x|x| = 0$  only for  $x = 0$ .

If  $p > 0$ : for  $x \leq 0$  the left-hand side is at most  $0 < p$ , so every root is positive. For  $x > 0$ ,  $x + 2p > 0$ , and the equation is  $x^2 + 2px = p$ . The function  $x^2 + 2px$  is increasing on  $(0, \infty)$ , takes the value 0 at  $x = 0$ , and grows beyond every bound, so it takes the value  $p$  exactly once: at  $x = -p + \sqrt{p^2 + p}$ .

If  $p < 0$ : write  $x = -u$  and  $p = -q$  with  $q > 0$ . The equation becomes  $-u|-u - 2q| = -q$ , that is,  $u|u + 2q| = q$ , which has exactly one root by the previous case.

8. Let  $a$ ,  $b$ , and  $c$  be the lengths of the sides of a triangle. Prove that

$$\frac{1}{3} \leq \frac{a^2 + b^2 + c^2}{(a + b + c)^2} < \frac{1}{2}$$

(15 points)

Solution: The left inequality is equivalent to  $(a + b + c)^2 \leq 3(a^2 + b^2 + c^2)$ , that is,

$$2(a^2 + b^2 + c^2) - 2(ab + bc + ca) = (a - b)^2 + (b - c)^2 + (c - a)^2 \geq 0$$

which is true (with equality for the equilateral triangle).

The right inequality is equivalent to  $2(a^2 + b^2 + c^2) < (a + b + c)^2$ , that is,  $a^2 + b^2 + c^2 < 2(ab + bc + ca)$ . By the triangle inequality,  $a < b + c$ , so  $a^2 < ab + ac$ ; similarly  $b^2 < ba + bc$  and  $c^2 < ca + cb$ . Adding these three inequalities gives exactly  $a^2 + b^2 + c^2 < 2(ab + bc + ca)$ .