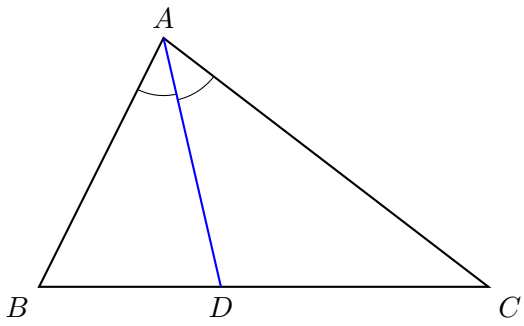
The **angle bisector** of an angle is the ray that divides the angle into two equal angles. In a triangle, the angle bisector from a vertex continues until it reaches the opposite side. In the picture below, AD is the angle bisector from vertex A in triangle ABC: the angles BAD and DAC are equal.



Where does the angle bisector meet the opposite side? If AB and AC have the same length, then by symmetry, D is the midpoint of BC. But if AB is shorter than AC, then D is closer to B than to C. The angle bisector theorem tells us exactly where D is.

The Angle Bisector Theorem. In triangle ABC, let AD be the angle bisector from vertex A, so that D is on side BC. Then

$$\frac{BD}{DC} = \frac{AB}{AC}.$$

In words: the angle bisector divides the opposite side in the same ratio as the other two sides of the triangle.

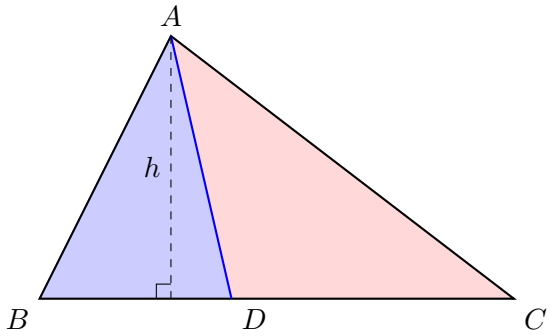
For example, if AB is half as long as AC, then BD is half as long as DC.

© Hidegkuti, 2026

The Proof

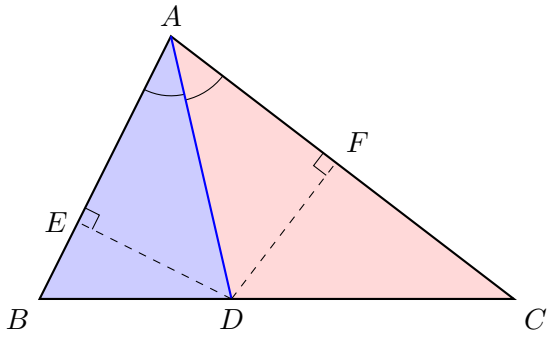
We will compute the ratio of the areas of the two smaller triangles, ABD and ADC, in two different ways. Let us denote the area of a triangle by T. Recall that the area of a triangle is half of the product of a side and the height that belongs to that side.

Step 1. Let us look at BD and DC as the bases of the two triangles. Both triangles have their third vertex at A, so they have the same height, h: the distance of point A from the line BC.



$$\frac{T_{ABD}}{T_{ADC}} = \frac{\frac{1}{2} \cdot BD \cdot h}{\frac{1}{2} \cdot DC \cdot h} = \frac{BD}{DC}$$

Step 2. Now let us look at AB and AC as the bases of the two triangles. The third vertex of both triangles is D. We drop perpendicular lines from D to the lines AB and AC, and call their feet E and F. Then DE is the height of triangle ABD that belongs to side AB, and DF is the height of triangle ADC that belongs to side AC.



The right triangles AED and AFD are congruent: they have a common hypotenuse, AD, and their angles at A are equal, because AD is an angle bisector. Therefore $DE = DF$. (In other words, every point of an angle bisector is at the same distance from the two sides of the angle.) Let us call this common distance d.

$$\frac{T_{ABD}}{T_{ADC}} = \frac{\frac{1}{2} \cdot AB \cdot d}{\frac{1}{2} \cdot AC \cdot d} = \frac{AB}{AC}$$

© Hidegkuti, 2026

Conclusion. Both $\frac{BD}{DC}$ and $\frac{AB}{AC}$ are equal to the same ratio of areas, $\frac{T_{ABD}}{T_{ADC}}$. So they are equal to each other, and $\frac{BD}{DC} = \frac{AB}{AC}$. ■

Using the Theorem

Example 1. The sides of triangle ABC are $AB = 6$, $AC = 9$, and $BC = 10$. The angle bisector from vertex A meets the opposite side, BC, at point D. Find the lengths of BD and DC.

Solution. By the angle bisector theorem, $\frac{BD}{DC} = \frac{6}{9} = \frac{2}{3}$. So we can write $BD = 2k$ and $DC = 3k$ for some number k. Together they make up side BC:

$$2k + 3k = 10 \qquad \Longrightarrow \qquad 5k = 10 \qquad \Longrightarrow \qquad k = 2$$

So $BD = 4$ and $DC = 6$. Check: $\frac{4}{6} = \frac{2}{3} = \frac{6}{9}$.

Example 2. In triangle ABC, $AB = 12$ and $AC = 8$. The angle bisector from vertex A meets the opposite side, BC, at point D, and $BD = 6$. Find DC.

Solution. We substitute into the theorem and solve the proportion.

$$\frac{6}{DC} = \frac{12}{8} \qquad \Longrightarrow \qquad 12 \cdot DC = 48 \qquad \Longrightarrow \qquad DC = 4$$

Example 3. In triangle ABC, $AB = 8$ and $AC = 12$. The angle bisector from vertex A meets the opposite side, BC, at point D, where $BD = x + 1$ and $DC = 2x - 1$. Find x, and the length of side BC.

Solution. By the angle bisector theorem,

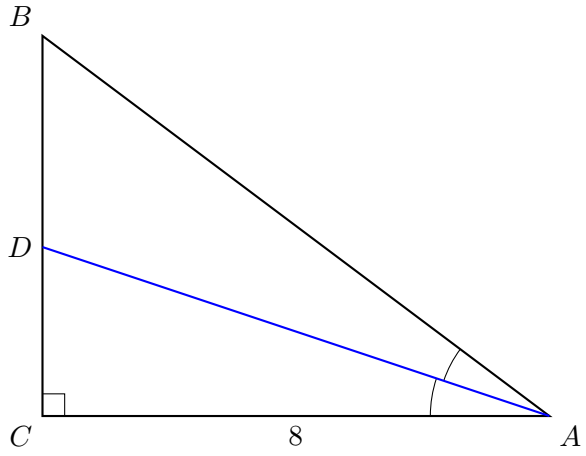
$$\frac{x+1}{2x-1} = \frac{8}{12} = \frac{2}{3}.$$

$$3\,(x+1) = 2\,(2x-1) \qquad \Longrightarrow \qquad 3x + 3 = 4x - 2 \qquad \Longrightarrow \qquad x = 5$$

So $BD = 6$, $DC = 9$, and $BC = 6 + 9 = 15$.

© Hidegkuti, 2026

Example 4. In the right triangle ABC, the right angle is at C, $AC = 8$, and $BC = 6$. The angle bisector from A meets side BC at D. Find the lengths of BD, DC, and AD.



Solution. First we find the hypotenuse with the Pythagorean theorem: $AB = \sqrt{8^2 + 6^2} = \sqrt{100} = 10$. By the angle bisector theorem,

$$\frac{BD}{DC} = \frac{AB}{AC} = \frac{10}{8} = \frac{5}{4}.$$

So $BD = 5k$ and $DC = 4k$, and $5k + 4k = 6$, which gives $k = \frac{2}{3}$. Therefore $BD = \frac{10}{3}$ and $DC = \frac{8}{3}$. Finally, ADC is a right triangle, so

$$AD = \sqrt{AC^2 + DC^2} = \sqrt{64 + \frac{64}{9}} = \sqrt{\frac{640}{9}} = \frac{8\sqrt{10}}{3} \approx 8.43.$$

© Hidegkuti, 2026



Practice Problems

1. In triangle ABC, the angle bisector from A meets side BC at D. In each case, find the missing lengths.
   a) $AB = 8$, $AC = 12$, $BC = 15$. Find BD and DC.
   b) $AB = 10$, $AC = 14$, $BD = 5$. Find DC.
   c) $AB = 9$, $BD = 6$, $DC = 4$. Find AC.
2. In triangle ABC, the angle bisector from A meets side BC at D. $AB = 10$, $AC = 16$, $BD = x$, and $DC = x + 3$. Find x, and the length of BC.
3. In the right triangle ABC, the right angle is at C, $AC = 12$, and $BC = 5$. The angle bisector from A meets side BC at D. Find the lengths of BD, DC, and AD. Round AD to two decimal places.
4. The angle bisector from A divides side BC of triangle ABC into two parts of lengths 6 and 8, with $BD = 6$. The sum of the other two sides is $AB + AC = 21$. Find AB, AC, and the perimeter of the triangle.
5. Use the angle bisector theorem to explain why, in an isosceles triangle, the angle bisector of the angle between the two equal sides meets the third side at its midpoint.
6. In Step 1 of the proof, why do the triangles ABD and ADC have the same height? Does this remain true if the angle at B is obtuse?

© Hidegkuti, 2026



Answers

1. a) $\frac{BD}{DC} = \frac{8}{12} = \frac{2}{3}$, so $BD = 2k$, $DC = 3k$, $5k = 15$: $BD = 6$, $DC = 9$

   b) $\frac{5}{DC} = \frac{10}{14}$, so $DC = 7$ c) $\frac{6}{4} = \frac{9}{AC}$, so $AC = 6$

2. $\frac{x}{x+3} = \frac{10}{16}$, so $16x = 10x + 30$, $x = 5$; $BD = 5$, $DC = 8$, $BC = 13$

3. $AB = 13$, $\frac{BD}{DC} = \frac{13}{12}$, so $13k + 12k = 5$, $k = \frac{1}{5}$: $BD = \frac{13}{5} = 2.6$, $DC = \frac{12}{5} = 2.4$, $AD = \sqrt{12^2 + 2.4^2} = \frac{12\sqrt{26}}{5} \approx 12.24$

4. $\frac{AB}{AC} = \frac{6}{8} = \frac{3}{4}$, so $AB = 3k$, $AC = 4k$, $7k = 21$, $k = 3$: $AB = 9$, $AC = 12$, and the perimeter is $9+12+14 = 35$.

5. If $AB = AC$, then $\frac{BD}{DC} = \frac{AB}{AC} = 1$, so $BD = DC$, and D is the midpoint of BC.

6. Both triangles have their third vertex at A, and their bases, BD and DC, are on the same line, BC. So both heights are the distance of A from the line BC. This remains true when the angle at B is obtuse; then the foot of the height is outside of the segment BC, but the height is still the same for both triangles.

© Hidegkuti, 2026