

Several kinds of angles can be drawn using a circle: the vertex may sit at the center, on the circle, inside the circle, or outside it. In every case the measure of the angle is determined by the arcs it cuts off. We will develop those relationships, beginning with a precise notion of arc measure.

Arcs and Their Measure

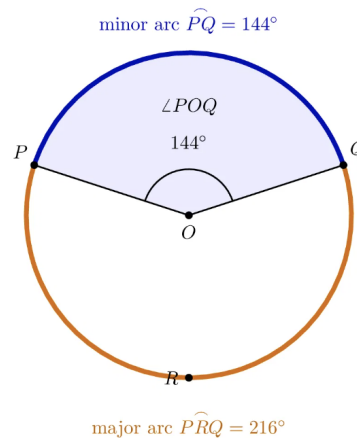
Two points A and B on a circle divide it into two arcs. If the arcs are unequal, the shorter is the **minor arc**, written $\widehat{AB}$, and the longer is the **major arc**, usually named with a third point on it, such as $\widehat{ACB}$. If A and B are the endpoints of a diameter, each arc is a **semicircle**.

Definition: (Degree measure of an arc) A full circle has arc measure 360° . The measure of a minor arc is defined to be the measure of its **central angle** (the angle with vertex at the center and sides through the endpoints of the arc). The measure of the corresponding major arc is 360° minus the measure of the minor arc, and the measure of a semicircle is 180° .

By definition, a central angle and its intercepted minor arc always have the same measure. This makes central angles the simplest case:

$$m(\angle AOB) = m\widehat{AB}.$$

Example 1. In a circle with center O , the central angle $\angle POQ$ measures 144° . Then $m\widehat{PQ} = 144^\circ$, and the major arc $\widehat{PRQ}$ measures $360^\circ - 144^\circ = 216^\circ$.



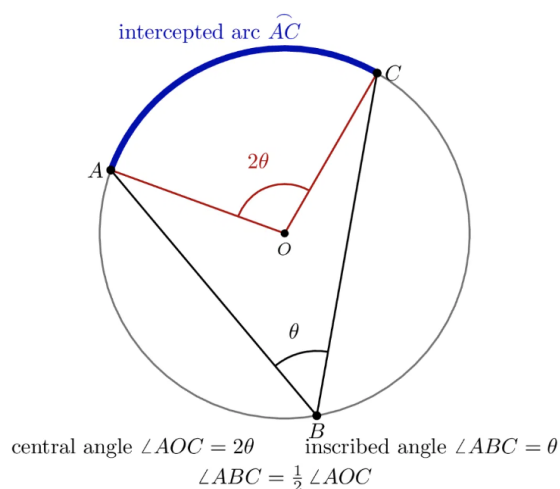
Inscribed Angles

Definition: An **inscribed angle** is an angle whose vertex lies on the circle and whose sides are chords of the circle. The arc lying in the interior of the angle is the **intercepted arc**.

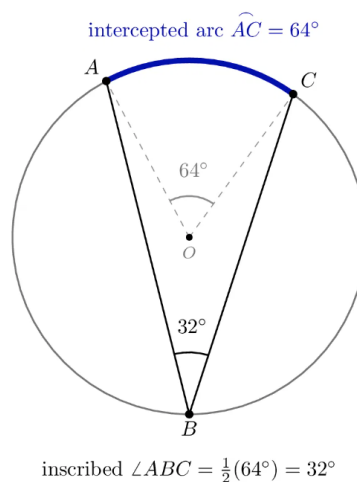
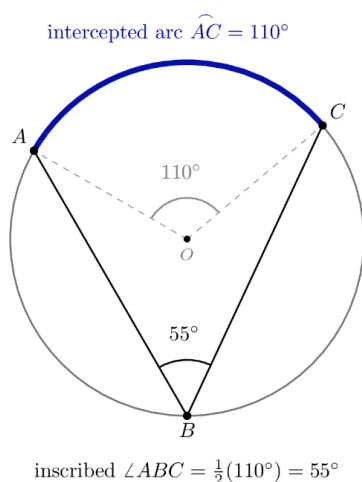
Theorem: (The Inscribed Angle Theorem) The measure of an inscribed angle is half the measure of its intercepted arc.

$$m(\angle ABC) = \frac{1}{2} m\widehat{AC}.$$

Equivalently, an inscribed angle is half of the central angle that intercepts the same arc.



Example 2. An inscribed angle intercepts an arc of 110° . The angle measures $\frac{1}{2}(110^\circ) = 55^\circ$. Conversely, an inscribed angle of 32° intercepts an arc of 64° .



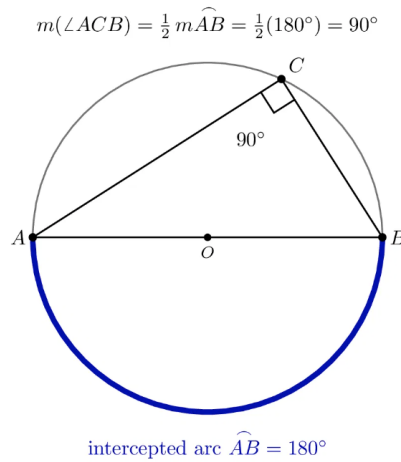
Consequences of the Inscribed Angle Theorem

Theorem: Two inscribed angles that intercept the same arc (or congruent arcs) are congruent.

Proof. Each equals half the measure of the common intercepted arc. □

There is an easy consequence of this theorem, also called Thales's Theorem.

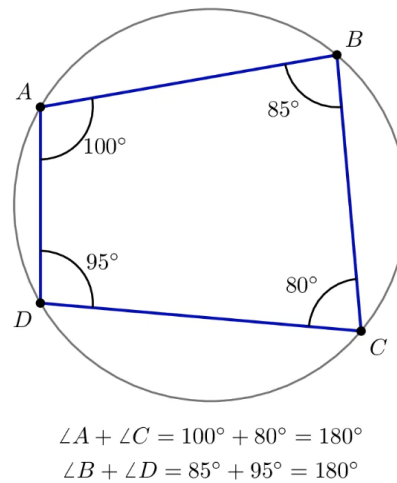
Theorem: An angle inscribed in a semicircle is a right angle. Equivalently, if $\overline{AB}$ is a diameter and C is any other point of the circle, then $m(\angle ACB) = 90^\circ$.



Proof. The intercepted arc $\widehat{AB}$ is a semicircle, of measure 180° , so $m(\angle ACB) = \frac{1}{2}(180^\circ) = 90^\circ$. □

Another interesting consequence is that opposite angles of a quadrilateral written into a circle add up to 180° .

Theorem: If a quadrilateral is inscribed in a circle, then each pair of opposite angles is supplementary.



Proof. Let $ABCD$ be inscribed in the circle. The inscribed angles $\angle A$ and $\angle C$ intercept the two arcs $\widehat{BCD}$ and $\widehat{DAB}$, which together make the whole circle. Hence

$$m(\angle A) + m(\angle C) = \frac{1}{2} m\widehat{BCD} + \frac{1}{2} m\widehat{DAB} = \frac{1}{2}(360^\circ) = 180^\circ.$$

The same argument applies to $\angle B$ and $\angle D$. □

Example 3. Quadrilateral $PQRS$ is inscribed in a circle. If $m(\angle P) = 84^\circ$ and $m(\angle Q) = 71^\circ$, then $m(\angle R) = 96^\circ$ and $m(\angle S) = 109^\circ$.

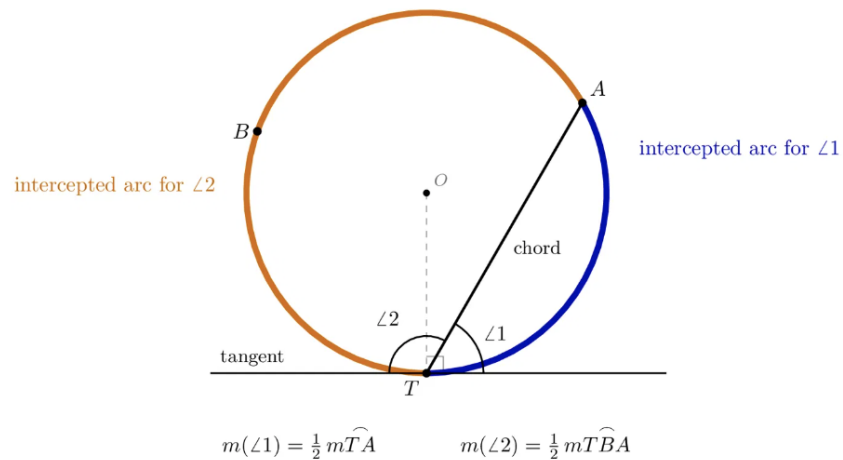
Tangent–Chord Angles

If a chord meets a tangent line at the point of tangency, the angle formed is neither central nor inscribed, but it obeys the same “half the arc” rule.

Theorem: (Tangent–Chord Angle) If a chord and a tangent meet at the point of tangency, then the measure of the angle formed equals half the measure of the intercepted arc:

$$m(\angle 1) = \frac{1}{2} m\widehat{AB},$$

where $\widehat{AB}$ is the arc lying inside the angle.



Example 4. A tangent and a chord meet at the point of tangency and intercept an arc of 140° . The tangent–chord angle measures 70° , and the angle on the other side of the chord measures $\frac{1}{2}(360^\circ - 140^\circ) = 110^\circ$.



Practice Problems

1. Points A, B, C lie on a circle with $m\widehat{AB} = 95^\circ$ and $m\widehat{BC} = 130^\circ$. Find $m\widehat{CA}$ and $m\widehat{ABC}$.
2. A central angle measures $x + 20$ degrees and intercepts an arc measuring $3x - 40$ degrees. Find x and the arc measure.
3. An inscribed angle intercepts an arc of 138° . Find the angle.
4. An inscribed angle measures 47° . Find its intercepted arc and the central angle on the same arc.
5. $\overline{AB}$ is a diameter of a circle and C is on the circle with $m\widehat{AC} = 116^\circ$. Find $m(\angle ABC)$, $m(\angle BAC)$, and $m(\angle ACB)$.
6. In a circle, inscribed angles $\angle ADB$ and $\angle ACB$ both intercept $\widehat{AB}$. If $m(\angle ADB) = 5x - 3$ and $m(\angle ACB) = 3x + 11$, find x and the measure of each angle.
7. Quadrilateral $WXYZ$ is inscribed in a circle with $m(\angle W) = 92^\circ$ and $m(\angle X) = 65^\circ$. Find $m(\angle Y)$ and $m(\angle Z)$.
8. A tangent and a chord meet at the point of tangency, intercepting an arc of 124° . Find both angles formed at that point.
9. A circle is divided by four points into arcs measuring $x, 2x, 3x$, and $4x$ degrees, in order around the circle. Find x , then find the measure of the inscribed angle that intercepts the arc of measure $3x$.
10. An inscribed angle and a central angle intercept the same arc. The central angle measures 80° less than three times the inscribed angle. Find both angle measures.



Answers

1. The three arcs $\widehat{AB}$, $\widehat{BC}$, $\widehat{CA}$ make a full circle, so $m\widehat{CA} = 360^\circ - 95^\circ - 130^\circ = 135^\circ$; and $m\widehat{ABC} = 95^\circ + 130^\circ = 225^\circ$.
2. A central angle equals its intercepted arc: $x + 20 = 3x - 40 \Rightarrow 2x = 60 \Rightarrow x = 30$. The arc (and the angle) measures $3(30) - 40 = 50^\circ$.
3. $\frac{1}{2}(138^\circ) = 69^\circ$.
4. Intercepted arc $= 2(47^\circ) = 94^\circ$; the central angle on the same arc also measures 94° .
5. $\overline{AB}$ is a diameter, so $\widehat{BC} = 180^\circ - 116^\circ = 64^\circ$.
 - $m(\angle ABC) = \frac{1}{2} m\widehat{AC} = \frac{1}{2}(116^\circ) = 58^\circ$.
 - $m(\angle BAC) = \frac{1}{2} m\widehat{BC} = \frac{1}{2}(64^\circ) = 32^\circ$.
 - $m(\angle ACB) = 90^\circ$ (angle in a semicircle).
6. Both angles intercept $\widehat{AB}$, so they are equal: $5x - 3 = 3x + 11 \Rightarrow 2x = 14 \Rightarrow x = 7$. Each angle measures $3(7) + 11 = 32^\circ$.
7. Opposite angles of a cyclic quadrilateral are supplementary: $m(\angle Y) = 180^\circ - 92^\circ = 88^\circ$ and $m(\angle Z) = 180^\circ - 65^\circ = 115^\circ$.
8. Tangent-chord angle $= \frac{1}{2}(124^\circ) = 62^\circ$; the angle on the other side of the chord intercepts the $360^\circ - 124^\circ = 236^\circ$ arc, so it measures $\frac{1}{2}(236^\circ) = 118^\circ$.
9. $x + 2x + 3x + 4x = 360^\circ \Rightarrow 10x = 360^\circ \Rightarrow x = 36^\circ$. The arc of measure $3x = 108^\circ$ is intercepted by an inscribed angle of $\frac{1}{2}(108^\circ) = 54^\circ$.
10. inscribed angle is 80° and the central angle is 160° .