

In geometry, to **construct** a figure means to draw it exactly, using only two tools: an unmarked **straightedge** and a **compass**. No ruler markings, no protractor, and no estimating by eye are allowed – every construction must be exact, and every step must be justified. This idea goes back over two thousand years to the ancient Greek mathematician Euclid, and it is still the cleanest way to understand why geometric relationships are actually true.

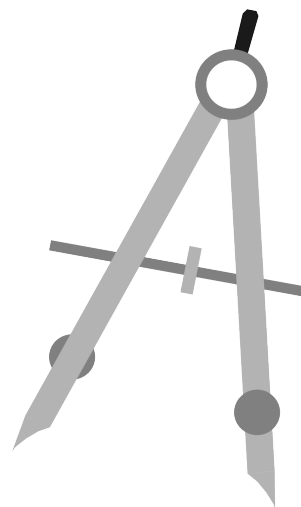
Part 1 – Tools and Rules

The straightedge. A straightedge lets us draw a straight line through two given points, or extend an existing segment. It has no markings on it – unlike a ruler, we are not allowed to measure a length with it. It can only be used to connect points we already know.

The compass. A compass draws a circle (or an arc of a circle) centered at a given point, with any radius we choose. Its most important use is to **transfer a length**: if we open the compass to match the distance between two points, we can move it anywhere and mark that same distance somewhere else, without ever writing down a number.



Straightedge
(should have no markings on it)



Compass

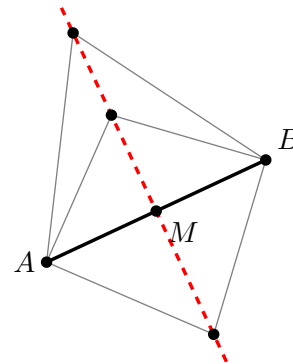
Every construction in this course must be built from only the following three steps, repeated as many times as needed.

1. Draw the line through two points already marked.
2. Draw the circle (or arc) with a given center already marked, passing through a given point already marked.
3. Mark a new point where two lines, two circles, or a line and a circle – all already drawn – intersect each other.

Nothing else is allowed: no measuring with a ruler, no protractor, and no guessing where a point "looks like" it should go.

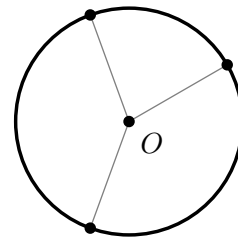
Part 2 – The Perpendicular Bisector

Definition: The **perpendicular bisector** of two points A and B is the set of all points in the plane that are equidistant from A and B .



The perpendicular bisector of A and B is a line, and that line is perpendicular to AB and passes through its midpoint.

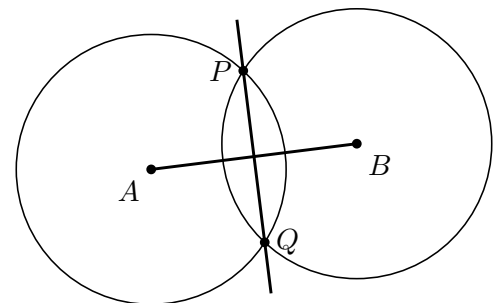
Definition: A **circle** is the set of all points in a plane that are a fixed distance (called the **radius**) from a fixed point (called the **center**).



Example 1. Construct the perpendicular bisector of line segment AB .

Construction:

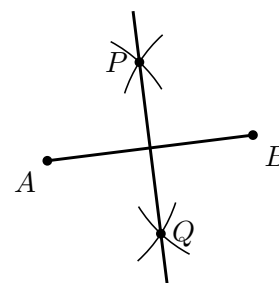
1. Open the compass to more than half the length of AB .
2. Draw a circle centered at A .
3. Without changing the compass width, draw a circle centered at B . Call the two intersection points P and Q .
4. Draw the line through P and Q .



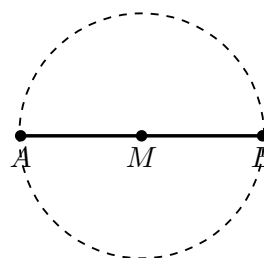
Line PQ is the perpendicular bisector of AB .

Why does this work? The two circles centered at A and B were drawn with the same compass width. So P , being on both the circle centered at A and the circle centered at B , is the same distance from A as it is from B ; the same is true of Q . By our definition, this means P and Q both lie on the perpendicular bisector of AB . Since two points uniquely determine a line, the line through P and Q must be the entire perpendicular bisector of AB .

Technically, a construction like this calls for two full circles, each centered at one of the two given points. In practice, though, we only ever draw the small arcs near where we expect them to intersect – this is just a habit construction diagrams follow to keep the picture uncluttered, and you will see it in every figure from here on.



Definition: The **Thales circle** of a line segment AB is the circle having AB as a diameter: its center is the midpoint of AB and its radius is half the length of AB .



Since we already know how to find the midpoint of a segment (using the perpendicular bisector), finding this circle is now easy.

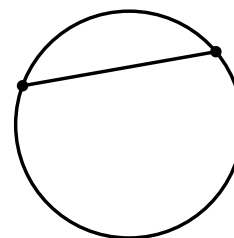
Example 2. Construct the Thales circle of line segment AB : the circle having AB as a diameter.

Construction:

1. Construct the perpendicular bisector of AB , and let M be the point where it crosses AB . (M is the midpoint of AB .)
2. Open the compass so that the point is on M and the pencil is on A .
3. With the point on M , draw the full circle.

This circle passes through both A and B and is the Thales circle of AB .

Definition: A **chord** of a circle is a segment that joins two points on the circle. (A diameter is simply the longest possible chord – one that passes through the center.)



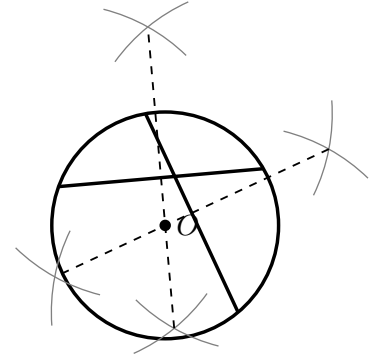
Chords give us a powerful tool: **the perpendicular bisector of any chord of a circle always passes through the center of that circle.** This makes sense, because every point on that perpendicular bisector is equidistant from the two endpoints of the chord – and the center, being the same distance (the radius) from every point on the circle, is one such point.

This gives us a way to locate the center of a circle, even when it is not marked: draw two chords that are not parallel to each other, construct their perpendicular bisectors, and the point where the two perpendicular bisectors cross is the center.

Example 3. The center of a given circle is not marked. Construct it.

Construction:

1. Draw any two chords of the circle, making sure they are not parallel to each other.
2. Construct the perpendicular bisector of each chord.
3. Mark the point where the two perpendicular bisectors cross. This point is the center of the circle.



Part 3 – Perpendiculars Through a Point

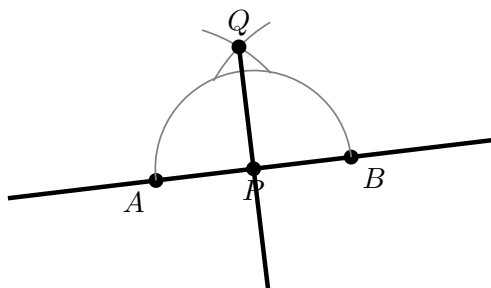
There are two situations in which we may need to construct a line perpendicular to a given line: the point we are given may lie **on** the line, or it may lie **off** the line. The two constructions are similar, but not quite the same.

Example 4. Point P lies on the given line. Construct the line through P perpendicular to the given line.

Construction:

1. With the point of the compass on P , draw an arc that crosses the line on both sides of P . Call the crossing points A and B . (So P is the midpoint of AB .)
2. Open the compass wider than before. With the point on A , draw an arc above the line; then, without changing the compass width, do the same centered at B . Call the point where these two arcs cross Q .
3. Draw the line through P and Q .

Line PQ is perpendicular to the given line at P .

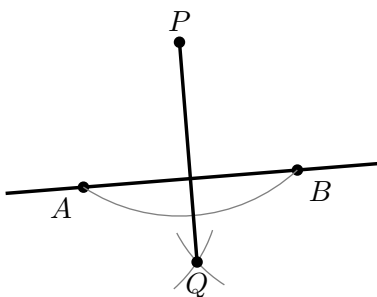


Example 5. Point P does *not* lie on the given line. Construct the line through P perpendicular to the given line.

Construction:

1. With the point of the compass on P , draw an arc that crosses the line at two points. Call them A and B .
2. With the point on A , draw an arc on the opposite side of the line from P ; then, without changing the compass width, do the same centered at B . Call the point where these two arcs cross Q .
3. Draw the line through P and Q .

Line PQ is perpendicular to the given line, and it crosses the given line at the foot of the perpendicular from P .



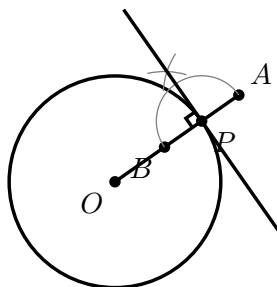
A line is **tangent** to a circle if it touches the circle at exactly one point. A key fact from geometry is that the tangent line at a point P on a circle is always perpendicular to the radius drawn to P . This means we already know how to construct it: just construct the perpendicular to the line through the center and P , at the point P .

Example 6. Point P lies on the given circle with center O . Construct the line tangent to the circle at P .

Construction:

1. Draw the radius OP .
2. Construct the line through P perpendicular to OP .

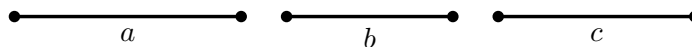
This perpendicular line is the tangent to the circle at P .



Part 4 – Constructing Triangles

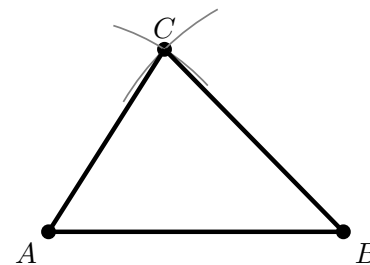
Given three segments, when can they be the three sides of a triangle? And if they can, how do we actually build that triangle with only a compass and straightedge – without knowing any of its angles?

Example 7. Construct a triangle whose three sides have the lengths of the segments a , b , and c shown below.



Construction:

1. Copy segment a ; call it AB .
2. Open the compass to the length of segment b . With the point on A , draw an arc.
3. Open the compass to the length of segment c . With the point on B , draw an arc crossing the one from step 2. Call the crossing point C .
4. Draw segments AC and BC .



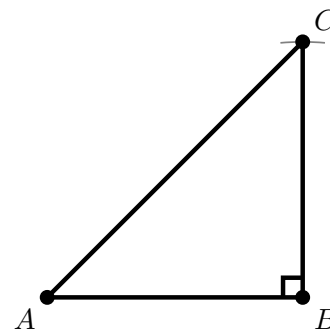
Triangle ABC has sides a , b , and c .

An isosceles right triangle has a right angle between its two equal legs. If we are given the length of a leg, this construction is a direct application of the perpendicular-through-a-point construction.

Example 8. Line segment AB is given. Construct an isosceles right triangle with legs of length AB .

Construction:

1. Construct the line through B perpendicular to AB .
2. Open the compass to the length AB . With the point on B , draw an arc crossing the perpendicular line. Call the crossing point C .
3. Draw segment AC .



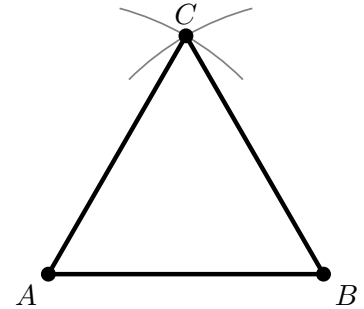
Triangle ABC is an isosceles right triangle, with the right angle at B and legs $AB \cong BC$.

An equilateral triangle has all three sides equal. If we are given one side, this construction is a direct application of building a triangle from three given sides – with all three sides equal to each other.

Example 9. Line segment AB is given. Construct an equilateral triangle with side AB .

Construction:

1. Open the compass to the length AB . With the point on A , draw an arc.
2. Without changing the compass width, place the point on B and draw another arc, crossing the first one. Call the crossing point C .
3. Draw segments AC and BC .



Triangle ABC is equilateral: $AB \cong BC \cong AC$.

Part 5 – Copying an Angle

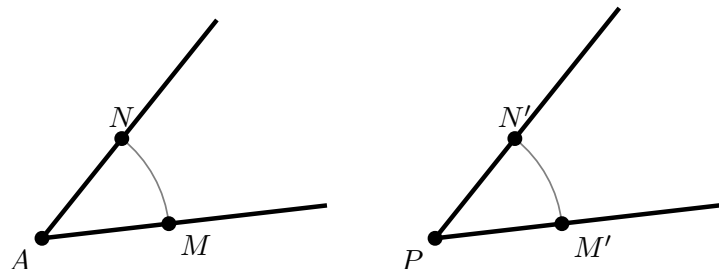
Just as we can copy a length without measuring it, we can copy an angle without measuring it in degrees – again, using only the compass to transfer information.

Example 10. Copy $\angle A$ so that its vertex is at point P .

Construction:

1. Draw a ray starting at P . (One side of the copied angle will lie along this ray.)
2. With the compass point on A , draw an arc that crosses both sides of $\angle A$. Call the crossing points M and N .
3. Without changing the compass width, place the point on P and draw a similar arc crossing the new ray. Call that crossing point M' .
4. Open the compass to the distance MN . Then, keeping that width, place the point on M' and mark a point N' where this new arc crosses the arc from step 3.
5. Draw the ray from P through N' .

Then $\angle M'PN' \cong \angle A$.



Part 6 – Tangents From an External Point

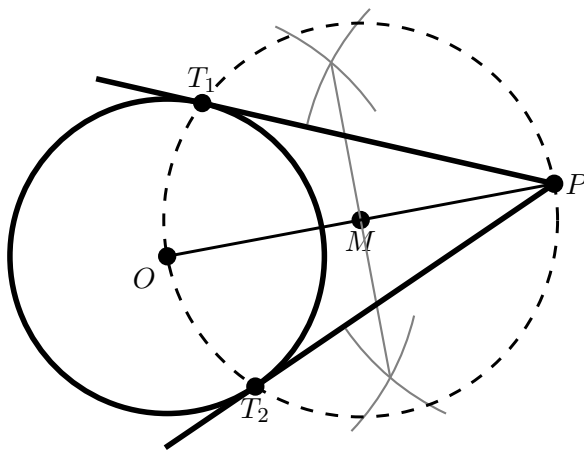
This last construction brings together two ideas we already know: the perpendicular bisector and the Thales circle. It relies on the fact that any angle inscribed in a semicircle – an angle whose vertex lies on a circle and whose sides pass through the two ends of a diameter – is a right angle. Since a tangent line touches a circle at a right angle to the radius, this gives us exactly what we need.

Example 11. Point P lies outside the given circle with center O . Construct the two tangent lines from P to the circle.

Construction:

1. Draw line segment OP .
2. Construct the perpendicular bisector of OP . Let M be the point where it crosses OP ; M is the midpoint of OP .
3. With the point on M , draw the Thales circle of OP : the circle through O and P centered at M .
4. This circle crosses the given circle at two points, T_1 and T_2 .
5. Draw lines PT_1 and PT_2 .

Lines PT_1 and PT_2 are the two tangent lines from P to the circle, and T_1, T_2 are the points of tangency.





Practice Problems

Using only a compass and straightedge, try each of the following constructions.

1. Draw a triangle. Construct the perpendicular bisector of all three sides. They should meet at a single point.
2. Draw a triangle. Construct all three altitudes – the perpendicular segment from each vertex to the line containing the opposite side. They should meet at a single point.
3. Draw a line segment. Construct a square having that segment as one side.
4. Draw a circle. Find its center, then inscribe a square in the circle: a square whose four vertices all lie on the circle.
5. Draw a triangle. Construct a second triangle, located elsewhere on the page, that is congruent to the first – that is, has the same three side lengths.