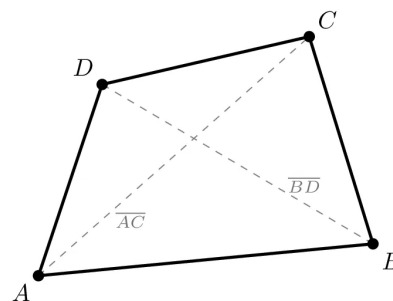


Quadrilaterals in General

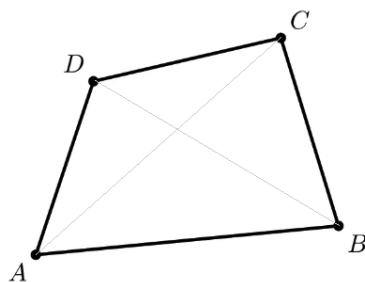
Definition: A **quadrilateral** is a polygon with four sides. We name it by its vertices in order around the figure, as in $ABCD$; its sides are then $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, $\overline{DA}$, and its **diagonals** are $\overline{AC}$ and $\overline{BD}$. Sides that share a vertex are **consecutive** (or **adjacent**); sides that do not are **opposite**. The same words apply to angles.



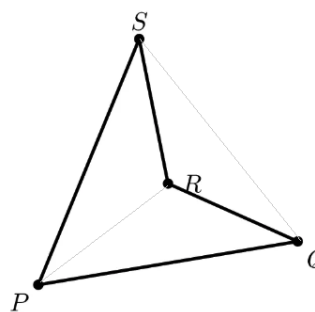
Theorem: The sum of the measures of the interior angles of any quadrilateral is 360° .

proof: A diagonal divides the quadrilateral into two triangles, each with angle sum 180° , and the four triangle angles at the two split vertices reassemble exactly into the quadrilateral's four angles. Hence the total is $2 \cdot 180^\circ = 360^\circ$.

Definition: A quadrilateral is **convex** if both diagonals lie in its interior; otherwise it is **concave**. Except where noted, the special quadrilaterals in this section are convex.



convex

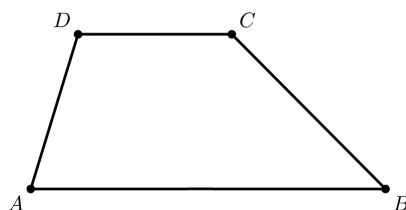


concave

The special families are obtained by imposing conditions of three kinds: *parallel sides* (trapezoid, parallelogram), *congruent sides* (rhombus, deltoid), and *right angles* (rectangle). A square satisfies conditions of all three kinds at once.

Trapezoids

Definition: A **trapezoid** is a quadrilateral with at least one pair of parallel sides. The parallel sides are the **bases**; the non-parallel sides are the **legs**. Two angles that share a base are a pair of **base angles**.



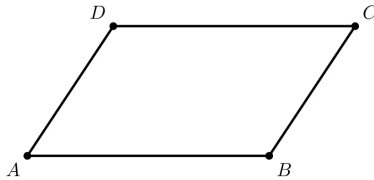
Basic Properties of Trapezoids:

Let $ABCD$ be a trapezoid with $\overline{AB} \parallel \overline{CD}$.

- Each leg is a transversal of the two parallel bases, so a pair of base angles on the same leg is **supplementary**:
 $m\angle A + m\angle D = 180^\circ$ and $m\angle B + m\angle C = 180^\circ$.

Parallelograms

Definition: A **parallelogram** is a quadrilateral with two pairs of parallel sides.

**Basic properties of parallelograms:**

In any parallelogram:

- opposite sides are congruent;
- opposite angles are congruent;
- consecutive angles are supplementary;
- the diagonals bisect each other;
- each diagonal divides it into two congruent triangles.

Theorem: A quadrilateral is a parallelogram if *any one* of the following holds:

- both pairs of opposite sides are parallel;
- both pairs of opposite sides are congruent;
- both pairs of opposite angles are congruent;
- one pair of opposite sides is both parallel and congruent;
- the diagonals bisect each other.

Rectangles

Definition: A **rectangle** is a parallelogram with a right angle.



Because consecutive angles of a parallelogram are supplementary and opposite angles are equal, one right angle forces all four: a rectangle has four right angles.

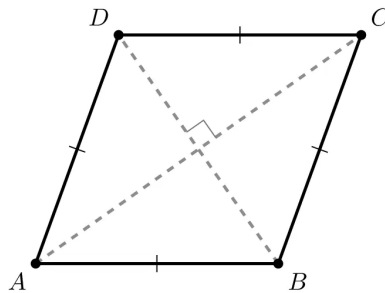
Basic properties of rectangles:

A rectangle has every parallelogram property, and in addition:

- all four angles are right angles;
- the diagonals are congruent (and, as in any parallelogram, bisect each other), so the diagonals of a rectangle bisect each other into four segments of equal length.

Rhombuses

Definition: A **rhombus** is a parallelogram with two congruent consecutive sides.



Since opposite sides of a parallelogram are congruent, one pair of congruent consecutive sides forces all four sides to be congruent: a rhombus is an equilateral quadrilateral that is also a parallelogram.

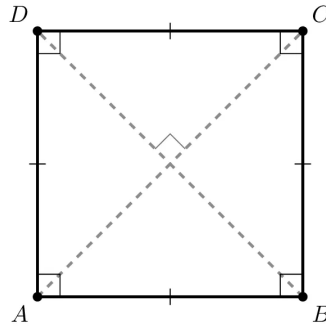
Basic properties of rhombuses:

A rhombus has every parallelogram property, and in addition:

- all four sides are congruent;
- the diagonals are perpendicular;
- each diagonal bisects a pair of opposite angles.

Squares

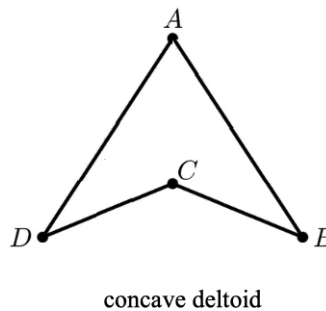
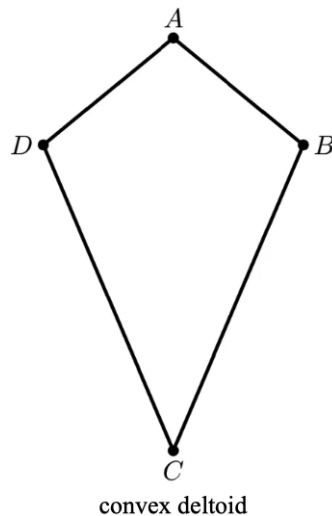
Definition: A **square** is a parallelogram that is both a rectangle and a rhombus; equivalently, a quadrilateral with four congruent sides and four right angles.



Basic Properties of squares: A square has *all* of the properties listed for parallelograms, rectangles, and rhombuses. In particular its diagonals are congruent, perpendicular, bisect each other, and bisect the angles (each into two 45° angles). A square is the **regular** quadrilateral. The diagonals of a square of side s each have length $s\sqrt{2}$, meet at the center, cross at 90° , and split the square into four congruent right isosceles triangles.

Deltoids (Kites)

Definition: A **deltoid** (also called a **kite**) is a quadrilateral with two distinct pairs of congruent consecutive sides. The vertex where each pair of congruent sides meets is a **vertex angle** of the kite; the other two angles are its **non-vertex angles**.



Basic properties of deltoids:

Let $ABCD$ be a kite with $AB = AD$ and $CB = CD$.

- The non-vertex angles are congruent: $m\angle B = m\angle D$.
- The diagonals are perpendicular.
- The diagonal through the vertex angles, $\overline{AC}$, is the perpendicular bisector of the other diagonal, and it bisects both vertex angles.
- Line AC is a line of symmetry (a kite has exactly one, unless it is also a rhombus).

Remark: If the two pairs of congruent sides are also congruent to *each other*, all four sides are equal and the kite is a **rhombus**; thus every rhombus is a special kite. A *non-convex* kite — one whose vertex C is pushed inside segment $\overline{BD}$'s side — is often called a **dart** or **arrowhead**. The perpendicular-diagonal and symmetry properties still hold for a dart.



Practice Problems

Identify. Give the most specific name for each quadrilateral.

1. Four right angles and four congruent sides.
2. Both pairs of opposite sides parallel; diagonals perpendicular; not all angles equal.
3. Two pairs of congruent consecutive sides; the two pairs are not congruent to each other.
4. Diagonals bisect each other and are congruent, but are not perpendicular.

True or false?

5. Every rectangle is a parallelogram.
6. Every parallelogram is a rectangle.
7. Every square is a rhombus.
8. Every rhombus is a square.
9. Every parallelogram is a trapezoid.
10. The diagonals of every kite bisect each other.



Answers

1. square
2. rhombus
3. deltoid
4. rectangle
5. true
6. false
7. true
8. false
9. true
10. false