Part 1 - The Same Old Equations with Fractions

We will further study solving linear equations. Let us first recall a few definitions.

> **Definition**: An **equation** is a statement in which two expressions (algebraic or numeric) are connected with an equal sign. A **solution** of an equation is a number (or an ordered set of numbers) that, when substituted into the variable(s) in the equation, makes the statement of equality of the equation true. To **solve an equation** is to find *all* solutions of it. The set of all solutions is also called the solution set.

For example, the equation $-x^2 + 3 = 4x - 2$ is an equation with two solutions, -5 and 1. We leave it to the reader to verify that these numbers are indeed solution. We will have to deploy systematic methods to find all solutions. The methods we will use usually depends on the type of equation. We have been studying the simplest equations, linear equations.

We have seen one- and two-step equations. First we will revisit those again, now that we have a greater number set. The most important thing to realize here is that the appearance of fractions does not change the fundamental methods of solving equations; rather, it makes each of the steps a bit more laborious. However, we should not let ourselves be intimidated by fractions.

Example 1. Solve each of the following equations.

$$\text{a) } 2x - 3 = 15 \qquad \text{b) } \frac{2}{3}x - \frac{1}{2} = -\frac{1}{3} \qquad \text{c) } \frac{x-4}{3} = -6 \qquad \text{d) } \frac{x + \frac{1}{2}}{\frac{3}{5}} = \frac{1}{3}$$

Solution: a) There is nothing new or unusual about this equation. The right-hand side is just a number. The unknown only appears on the left-hand side. There, we see two operations: the unknown was first multiplied by 2 and then 3 was subtracted. To isolate the unknown on the left-hand side, we will perform the inverse operations to both sides, in a reverse order. We will first add 3 and then we will divide by 2.

$$
\begin{aligned}
2x - 3 &= 15 \qquad \text{add 3} \\
2x &= 18 \qquad \text{divide by 2} \\
x &= 9
\end{aligned}
$$

We check: If $x = 9$, then the left-hand side is: $\quad \text{LHS} = 2x - 3 = 2 \cdot 9 - 3 = 18 - 3 = 15 = \text{RHS}$

Thus our solution, $x = \boxed{9}$ is correct.

b) Consider the equation $\frac{2}{3}x - \frac{1}{2} = -\frac{1}{3}$. There is nothing new or unusual about this equation either. If we could solve $2x - 3 = 15$, then we can solve this equation using the same steps. On the left-hand side, the unknown was multiplied by $\frac{2}{3}$ and then $\frac{1}{2}$ was subtracted. To isolate the unknown, we will perform to both sides the inverse operations, in a reverse order. This means that we will add $\frac{1}{2}$ and then divide by $\frac{2}{3}$. The main computation should be clean; we should just record the result of each step. We will perform computations on the margin.

$$\frac{2}{3}x - \frac{1}{2} = -\frac{1}{3} \qquad \text{add } \frac{1}{2} \qquad\qquad \text{margin work: } \quad -\frac{1}{3} + \frac{1}{2} = \frac{-2}{6} + \frac{3}{6} = \frac{1}{6}$$

$$\frac{2}{3}x = \frac{1}{6} \qquad \text{divide by } \frac{2}{3} \qquad\qquad \frac{1}{6} \div \frac{2}{3} = \frac{1}{6} \cdot \frac{3}{2} = \frac{1}{2 \cdot 3} \cdot \frac{\cancel{3}}{2} = \frac{1}{4}$$

$$x = \frac{1}{4}$$

We check: If $x = \frac{1}{4}$, then the left-hand side is

$$\text{LHS} = \frac{2}{3}\left(\frac{1}{4}\right) - \frac{1}{2} = \frac{2}{3} \cdot \frac{1}{2 \cdot 2} - \frac{1}{2} = \frac{1}{6} - \frac{1}{2} = \frac{1}{6} - \frac{3}{6} = \frac{-2}{6} = -\frac{1}{3} = \text{RHS}$$

Thus our solution, $x = \boxed{\dfrac{1}{4}}$ is correct.

c) Consider now the equation $\dfrac{x-4}{3} = -6$. This is a simple two-step equation, we only have it here to serve as an analogous example for part d. On the left-hand side we have first a subtraction of 4 and then a division by 3. To isolate the unknown, we will multiply by 3 and then add 4. As always, we wil perform all operations to both sides.

$$\frac{x-4}{3} = -6 \qquad \text{multiply by 3}$$
$$x - 4 = -18 \qquad \text{add } 4$$
$$x = -14$$

We check: If $x = -14$, then the left-hand side is: $\text{LHS} = \dfrac{x-4}{3} = \dfrac{-14-4}{3} = \dfrac{-18}{3} = -6 = \text{RHS}$

Thus our solution, $x = \boxed{-14}$ is correct.

d) Consider now the equation $\dfrac{x + \frac{1}{2}}{\frac{3}{5}} = \dfrac{1}{3}$. If we could solve $\dfrac{x-4}{3} = -6$, then we can solve this equation using the same steps. On the left-hand side, there was an addition of $\dfrac{1}{2}$, and then a division by $\dfrac{3}{5}$. To isolate the unknown, we will perform to both sides the inverse operations, in a reverse order. This means that we will multiply by $\dfrac{3}{5}$ and then subtract $\dfrac{1}{2}$. The main computation should be clean; we should just record the result of each step. We will perform computations on the margin.

$$\frac{x + \frac{1}{2}}{\frac{3}{5}} = \frac{1}{3} \qquad \text{multiply by } \frac{3}{5} \qquad\qquad \text{margin work: } \quad \frac{1}{3} \cdot \frac{3}{5} = \frac{1}{5}$$

$$x + \frac{1}{2} = \frac{1}{5} \qquad \text{subtract } \frac{1}{2} \qquad\qquad \frac{1}{5} - \frac{1}{2} = \frac{2-5}{10} = -\frac{3}{10}$$

$$x = -\frac{3}{10}$$

We check: If $x = -\dfrac{3}{10}$, then the left-hand side is

$$\text{LHS} = \dfrac{x + \dfrac{1}{2}}{\dfrac{3}{5}} = \dfrac{-\dfrac{3}{10} + \dfrac{1}{2}}{\dfrac{3}{5}} = \dfrac{\dfrac{-3+5}{10}}{\dfrac{3}{5}} = \dfrac{\dfrac{2}{10}}{\dfrac{3}{5}} = \dfrac{\dfrac{1}{5}}{\dfrac{3}{5}} = \dfrac{1}{5} \cdot \dfrac{5}{3} = \dfrac{1}{3} = \text{RHS}$$

Thus our solution, $x = \boxed{-\dfrac{3}{10}}$ is correct.

Part 2 - The Unknown is On Both Sides

We will now consider slightly more complex situations, in which the unknown appears on both sides of the equation.

Example 2. Solve the equation $2x - 8 = 5x + 10$. Make sure to check your solution.

Solution: Recall that in an algebraic expression, the numbers multiplying the unknown are called **coefficients**. In this equation, the coefficients of x are 2 on the left-hand side, and 5 on the rigth-hand side. We can easily reduce this equation to a two-step equation by subtracting $2x$ from both sides.

$$
\begin{aligned}
2x - 8 &= 5x + 10 \qquad &&\text{subtract } 2x \\
-8 &= 3x + 10 \qquad &&\text{subtract } 10 \\
-18 &= 3x \qquad &&\text{divide by } 3 \\
-6 &= x
\end{aligned}
$$

The only solution of this equation is -6. We check; if $x = -6$, then the left-hand side (LHS) is

$$\text{LHS} = 2(-6) - 8 = -12 - 8 = -20$$

and the right-hand side (RHS) is

$$\text{RHS} = 5(-6) + 10 = -30 + 10 = -20$$

So our solution, $\boxed{x = -6}$ is correct.

Note that there is another way to reduce this equation to a two-step equation. Instead of subtracting $2x$, we could also subtract $5x$. Both methods lead to a correct solution. This is a new situation for us. We have a choice between two methods. Although both methods are equally correct, one is better than the other one because it makes the rest of the computation easier. These kind of strategic decisions will become more and more important as we advance in mathematics.

Let us see now, what would happen if we chose to subtract $5x$.

$$
\begin{aligned}
2x - 8 &= 5x + 10 \qquad &&\text{subtract } 5x \\
-3x - 8 &= 10 \qquad &&\text{add } 8 \\
-3x &= 18 \qquad &&\text{divide by } -3 \\
x &= -6
\end{aligned}
$$

As we can see, now we had to divide by a negative number in the last step. This can be always avoided if we address the side on which the unknown has the smaller coefficient. Keep in mind, coefficients include the sign.

Example 3. Solve the equation $7a - 12 = -a + 20$. Make sure to check your solution.

Solution: Let us first compare the coefficients. On the left-hand side, the coefficient of a is 7. On the right-hand side, the coefficient of a is -1. Since -1 is less than 7, we will eliminate $-a$ from the right-hand side. We reduce this equation to a two-step equation by adding a to both sides.

$$\begin{aligned}
7a - 12 &= -a + 20 & \text{add } a \\
8a - 12 &= 20 & \text{add } 12 \\
8a &= 32 & \text{divide by } 8 \\
a &= 4
\end{aligned}$$

So the only solution of this equation is 4. We check; if $a = 4$,

$$\text{LHS} = 7 \cdot 4 - 12 = 28 - 12 = 16 \text{ and } \text{RHS} = -4 + 20 = 16 \implies \text{LHS} = \text{RHS}$$

So our solution, $\boxed{a = 4}$ is correct.

Example 4. Solve the equation $-4x + 2 = -x + 17$. Make sure to check your solutions.

Solution: -4 is less than -1. Therefore, we will eliminate $-4x$ from the left-hand side.

$$\begin{aligned}
-4x + 2 &= -x + 17 & \text{add } 4x \\
2 &= 3x + 17 & \text{subtract } 17 \\
-15 &= 3x & \text{divide by } 3 \\
-5 &= x
\end{aligned}$$

We check; if $x = -5$, then

$$\text{LHS} = -4(-5) + 2 = 20 + 2 = 22 \text{ and } \text{RHS} = -(-5) + 17 = 5 + 17 = 22 \implies \text{LHS} = \text{RHS}$$

So our solution, $\boxed{x = -5}$ is correct.

Example 5. Solve the equation $\dfrac{1}{2}m - 1 = \dfrac{5}{4}m - \dfrac{1}{4}$. Make sure to check your solutions.

$$\begin{aligned}
\frac{1}{2}m - 1 &= \frac{5}{4}m - \frac{1}{4} & \text{subtract } \frac{1}{2}m \\
-1 &= \frac{3}{4}m - \frac{1}{4} & \text{add } \frac{1}{4} \\
-\frac{3}{4} &= \frac{3}{4}m & \text{divide by } \frac{3}{4} \\
-1 &= m
\end{aligned}$$

margin work: $\dfrac{5}{4} - \dfrac{1}{2} = \dfrac{5}{4} - \dfrac{2}{4} = \dfrac{3}{4}$

$-1 + \dfrac{1}{4} = \dfrac{-4}{4} + \dfrac{1}{4} = -\dfrac{3}{4}$

$-\dfrac{3}{4} \div \dfrac{3}{4} = -\dfrac{3}{4} \cdot \dfrac{4}{3} = -1$

So the only solution of this equation is -1. We check; if $m = -1$,

$$\text{LHS} = \frac{1}{2}(-1) - 1 = -\frac{1}{2} - 1 = \frac{-1}{2} - \frac{2}{2} = -\frac{3}{2} \text{ and}$$

$$\text{RHS} = \frac{5}{4}(-1) - \frac{1}{4} = -\frac{5}{4} - \frac{1}{4} = -\frac{6}{4} = -\frac{3}{2} \implies \text{LHS} = \text{RHS}$$

So our solution, $\boxed{m = -1}$ is correct.

Again, the steps were the same as before, they only took a bit more work to perform because of the appearance of fractions.

Once the unknown appears on both sides, we might face new situations that were not possible in two-step equations. Consider each of the following.

Example 6. Solve each of the following equations. Make sure to check your solutions.

      a) $5y + 16 = 5y + 16$ b) $-6x + 5 = -6x + 9$

Solution: a) $5y + 16 = 5y + 16$

This equation looks different from all the others because the two sides are identical. Logically, the value of two sides will be equal no matter what number we substitute into the equation. Computation will confirm this idea.

$$\begin{aligned} 5y + 16 &= 5y + 16 &&\text{subtract } 5y \\ 16 &= 16 \end{aligned}$$

The statement $16 = 16$ is true, no matter what the value of y is. Such a statement is called an **unconditionally true statement** or **identity**. All numbers are solution of this equation.

  b) $-6x + 5 = -6x + 9$

$$\begin{aligned} -6x + 5 &= -6x + 9 &&\text{add } 6x \\ 5 &= 9 \end{aligned}$$

When we tried to eliminate the unknown from one side, it disappeared from both sides. We are left with the statement $5 = 9$. No matter what the value of the unknown is, this statement can not be made true. Indeed, our last line is an **unconditionally false statement**. This means that there is no number that could make this statement true, and so this equation **has no solution**. An equation like this is called a **contradiction**.

These strange situations happen when the unknown has the same coefficient on both sides. Otherwise, there is exactly one solution. Based on their solution sets, linear equations can be classified as follows.

---

1. If the last line is of the form $x = 5$, the equation is called **conditional**.
   This is because the truth value of the statement depends on the value of x. True if x is 5, false otherwise. A conditional equation has exactly **one solution**.

2. If the last line is of the form $1 = 1$, the equation is unconditionally true. Such an equation is called an **identity** and **all numbers are solution**s of it.

3. If the last line is of the form $3 = 8$, the equation is unconditionally false. Such an equation is called a **contradiction** and its solution set is **the empty set**.

Discussion: Classify each of the given equations as a conditional equation, an identity, or a contradiction.

    a) $3x + 1 = 3x - 1$ b) $2x - 4 = 7x - 4$ c) $x - 4 = 4 - x$ d) $x - 1 = -1 + x$

Part 3 - Using the Distributive Law

Linear equations might be more complicated. Most often we will be dealing with the distributive law to eliminate parentheses. Then we combine like terms on both sides. After that, these equations will be reduced to the type we just saw.

Example 7. Solve each of the given equations. Make sure to check your solutions.

a) $3x - 2(4 - x) = 3(3x - 1) - (x - 7)$ b) $4(y - 2) - 6(3y - 5) = 5 - 2(7y + 1)$ c) $\frac{2}{3}x - 4 - \frac{1}{6}(x + 6) = \frac{1}{2}(x - 10)$

Solution: a) We first eliminate the parentheses by applying the distributive law.

$$
\begin{array}{rcll}
3x - 2(4 - x) & = & 3(3x - 1) - (x - 7) & \text{eliminate parentheses} \qquad \text{Caution!} \quad -2(-x) = 2x \\
3x - 8 + 2x & = & 9x - 3 - x + 7 & \text{combine like terms} \qquad\qquad\qquad \text{and} \ -(-7) = 7 \\
5x - 8 & = & 8x + 4 & \text{subtract } 5x \\
-8 & = & 3x + 4 & \text{subtract } 4 \\
-12 & = & 3x & \text{divide by } 3 \\
-4 & = & x &
\end{array}
$$

We check: if $x = -4$, then

$$
\begin{aligned}
\text{LHS} & = 3(-4) - 2(4 - (-4)) = 3(-4) - 2 \cdot 8 = -12 - 16 = -28 \quad \text{and} \\
\text{RHS} & = 3(3(-4) - 1) - (-4 - 7) = 3(-12 - 1) - (-11) = 3(-13) + 11 = -39 + 11 = -28 \quad \Longrightarrow \quad \text{LHS} = \text{RHS}
\end{aligned}
$$

So our solution, $\boxed{x = -4}$ is correct.

b) We first eliminate the parentheses by applying the distributive law.

$$
\begin{array}{rcll}
4(y - 2) - 6(3y - 5) & = & 5 - 2(7y + 1) & \text{eliminate parentheses} \\
4y - 8 - 18y + 30 & = & 5 - 14y - 2 & \text{combine like terms} \\
-14y + 22 & = & -14y + 3 & \text{add } 14y \\
22 & = & 3 &
\end{array}
$$

We are left with the statement $22 = 3$. No matter what the value of the unknown is, this statement can not be made true, this is an **unconditionally false statement**. This equation **has no solution**. An equation like this is called a **contradiction**.

c) We first eliminate the parentheses by applying the distributive law.

$$
\begin{array}{rcll}
\frac{2}{3}x - 4 - \frac{1}{6}(x + 6) & = & \frac{1}{2}(x - 10) & \text{eliminate parentheses} \\
\frac{2}{3}x - 4 - \frac{1}{6}x - 1 & = & \frac{1}{2}x - 5 & \text{combine like terms} \qquad \text{margin work: } \ \frac{2}{3} - \frac{1}{6} = \frac{4 - 1}{6} = \frac{3}{6} = \frac{1}{2} \\
\frac{1}{2}x - 5 & = & \frac{1}{2}x - 5 & \text{subtract } \frac{1}{2}x \\
-5 & = & -5 &
\end{array}
$$

When we tried to eliminate the unknown from one side, it disappeared again from both sides. We are left with the statement $-5 = -5$. This statement is true for all values of x. Indeed, our last line is an **unconditionally true statement**.

This means that every number makes make this statement true, and so the solution set of this equation is the set of all numbers. An equation like this is called an **identity**.

We often use identities in mathematics, although it seems at first that we would not need equations whose solution set is every number. Consider the following equation: $a + b = b + a$. This equation is an identity, because every pair of numbers is a solution. We use this identity to express a property of *addition*: that the sum of two numbers does not depend on the order of the two numbers.

Sample Problems

Solve each of the following equations. Make sure to check your solutions.

1. $2x + 3 = 4x + 9$

2. $3w - 5 = 5(w + 1)$

3. $7x - 2 = 5x - 2$

4. $3y - 9 = -2y + 4$

5. $2w + 1 = 2w - 9$

6. $\frac{1}{6}x - 1 = \frac{2}{3}x + 4$

7. $-\frac{2}{5}x + \frac{1}{3} = \frac{4}{15}x$

8. $4 - x = 3(x - 7)$

9. $7(j - 5) + 9 = 2(-2j + 5) + 5j$

10. $3(x - 5) - 5(x - 1) = -2x + 1$

11. $\frac{2}{3}(x - 7) = \frac{4}{5}(x + 1)$

Practice Problems

Solve each of the following equations. Make sure to check your solutions.

1. $5x - 3 = x + 9$

2. $-x + 13 = 2x + 1$

3. $-2x + 4 = 5x - 10$

4. $5a + 1 = -3a - 1$

5. $5x - 7 = 6x + 8$

6. $8x - 1 = 3x + 19$

7. $-7x - 1 = 3x - 21$

8. $2 - 3(y - 1) = 2y - 7$

9. $3(x - 4) = 2(x + 5)$

10. $4(5x + 1) = 6x + 4$

11. $a - 3 = 5(a - 1) - 2$

12. $4m - 1 = -4m + 3$

13. $3y - 2 = -2y + 18$

14. $8(x - 3) - 3(5 - 2x) = x$

15. $5(x - 1) - 3(x + 1) = 3x - 8$

16. $\frac{2}{3}x - 1 = -\frac{2}{3}\left(x + \frac{1}{2}\right)$

17. $-2x - (3x - 1) = 2(5 - 3x)$

18. $3(x - 4) + 5(x + 8) = 2(x - 1)$

19. $5(x - 1) - 3(-x + 1) = -3 + 8x$

20. $\frac{3}{4}x - \frac{2}{3}\left(x + \frac{1}{2}\right) = -x$

21. $\frac{3}{8}x + 1\frac{4}{5} = \frac{1}{4}x + 1\frac{3}{10}$

22. $\frac{1}{2}\left(x + \frac{2}{3}\right) - \frac{1}{3}\left(x - \frac{1}{2}\right) = \frac{1}{6}x + \frac{1}{2}$

23. $2(b + 1) - 5(b - 3) = 2(b - 7) + 1$

24. $-3y - 2(5y - 1) = 7y - 3$

25. $3(2x - 1) - 5(2 - x) = 4(x - 1) + 5$

26. $3(2x - 7) - 2(5x + 2) = -5x - 30$

27. $3(x - 4) - 4(x - 3) = 3(x - 2) + 2(3 - x)$

28. $\frac{1}{2}(x + 1) - \frac{3}{5}(x - 1) = -\frac{3}{8}x$

Answers

Discussion

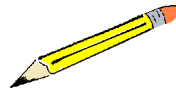a) contradiction   b) conditional   c) conditional   d) identity

Sample Problems

1. -3 2. -5 3. 0 4. $\dfrac{13}{5}$ 5. There is no solution.   6. -10 7. $\dfrac{1}{2}$ 8. $\dfrac{25}{4}$ 9. 6 10. There is no solution
11. -41

Practice Problems

1. 3 2. 4 3. 2 4. $-\dfrac{1}{4}$ 5. -15 6. 4 7. 2 8. $\dfrac{12}{5}$ 9. 22 10. 0 11. 1 12. $\dfrac{1}{2}$ 13. 4 14. 3

15. 0 16. $\dfrac{1}{2}$ 17. 9 18. -5 19. no solution   20. $\dfrac{4}{13}$ 21. -4 22. all numbers are solution   23. 6

24. $\dfrac{1}{4}$ 25. 2 26. -5 27. 0 28. -4

Sample Problems   Solutions

1. $2x + 3 = 4x + 9$

   Solution:

$$
\begin{aligned}
2x + 3 &= 4x + 9 &&\text{subtract } 2x \text{ from both sides} \\
3 &= 2x + 9 &&\text{subtract } 9 \text{ from both sides} \\
-6 &= 2x &&\text{divide both sides by 2} \\
-3 &= x
\end{aligned}
$$

We check: if $x = -3$, then

$$
\begin{aligned}
\text{LHS} &= 2(-3) + 3 = -6 + 3 = -3 \\
\text{RHS} &= 4(-3) + 9 = -12 + 9 = -3
\end{aligned}
$$

Thus our solution, $\boxed{x = -3}$ is correct. (Note: LHS is short for the left-hand side and RHS is short for the right-hand side.)

2. $3w - 5 = 5(w + 1)$

   Solution: we first apply the law of distributivity to simplify the right-hand side.

$$
\begin{aligned}
3w - 5 &= 5(w + 1) \\
3w - 5 &= 5w + 5 &&\text{subtract } 3w \text{ from both sides} \\
-5 &= 2w + 5 &&\text{subtract } 5 \text{ from both sides} \\
-10 &= 2w &&\text{divide both sides by 2} \\
-5 &= w
\end{aligned}
$$

We check. If $w = -5$, then

$$
\begin{aligned}
\text{LHS} &= 3(-5) - 5 = -15 - 5 = -20 \\
\text{RHS} &= 5((-5) + 1) = 5(-4) = -20
\end{aligned}
$$

Thus our solution, $\boxed{w = -5}$ is correct.

3. $7x - 2 = 5x - 2$

   Solution:

$$
\begin{aligned}
7x - 2 &= 5x - 2 &&\text{subtract } 5x \text{ from both sides} \\
2x - 2 &= -2 &&\text{add } 2 \text{ to both sides} \\
2x &= 0 &&\text{divide both sides by 2} \\
x &= 0
\end{aligned}
$$

We check: if $x = 0$, then

$$\text{LHS} = 7(0) - 2 = 0 - 2 = -2$$
$$\text{RHS} = 5(0) - 2 = 0 - 2 = -2$$

Thus our solution, $\boxed{x = 0}$ is correct.

4. $3y - 9 = -2y + 4$

   Solution:

$$
\begin{aligned}
3y - 9 &= -2y + 4 && \text{add } 2y \text{ to both sides} \\
5y - 9 &= 4 && \text{add } 9 \text{ to both sides} \\
5y &= 13 && \text{divide both sides by } 5 \\
y &= \frac{13}{5}
\end{aligned}
$$

We check. If $y = \dfrac{13}{5}$, then

$$\text{LHS} = 3\left(\frac{13}{5}\right) - 9 = \frac{3}{1} \cdot \frac{13}{5} - 9 = \frac{39}{5} - \frac{9}{1} = \frac{39}{5} - \frac{45}{5} = \frac{-6}{5} = -\frac{6}{5}$$
$$\text{RHS} = -2\left(\frac{13}{5}\right) + 4 = \frac{-2}{1} \cdot \frac{13}{5} + \frac{4}{1} = \frac{-26}{5} + \frac{20}{5} = \frac{-6}{5} = -\frac{6}{5}$$

Thus $\boxed{y = \dfrac{13}{5}}$ is the correct solution.

5. $2w + 1 = 2w - 9$

   Solution:

$$
\begin{aligned}
2w + 1 &= 2w - 9 && \text{subtract } 2w \text{ from both sides} \\
1 &= -9
\end{aligned}
$$

The statement $1 = -9$ is false no matter what the value of w is. Such a statement is called an **unconditionally false statement**, or **contradiction**. **This equation has** $\boxed{\text{no solution}}$.

6. $\dfrac{1}{2}x - 1 = \dfrac{2}{3}x + 4$

   Solution: Structurally, this equation is no different from the previous equations. However, because the coefficients of x are fractions, each step will take a bit more work.

$$
\begin{aligned}
\frac{1}{2}x - 1 &= \frac{2}{3}x + 4 && \text{subtract } \frac{1}{2}x \text{ from both sides} \\
-1 &= \frac{1}{6}x + 4 && \text{subtract 4 from both sides} \\
-5 &= \frac{1}{6}x && \text{divide both sides by } \frac{1}{6} \\
-30 &= x
\end{aligned}
$$

Here are the computations for each step. To subtract $\frac{1}{2}x$ from the right-hand side:

$$\frac{2}{3}x - \frac{1}{2}x = \left(\frac{2}{3} - \frac{1}{2}\right)x = \left(\frac{4}{6} - \frac{3}{6}\right)x = \frac{4-3}{6}x = \frac{1}{6}x$$

We divide both sides by $\frac{1}{6}$. To divide is to multiply by the reciprocal:

$$-5 \div \frac{1}{6} = \frac{-5}{1} \div \frac{1}{6} = \frac{-5}{1} \cdot \frac{6}{1} = \frac{-30}{1} = -30$$

We check: if $x = -30$, then

$$\begin{aligned}
\text{LHS} &= \frac{1}{2}(-30) - 1 = -15 - 1 = -16 \\
\text{RHS} &= \frac{2}{3}(-30) + 4 = \frac{2}{3} \cdot \frac{-30}{1} + 4 = \frac{-60}{3} + 4 = -20 + 4 = -16
\end{aligned}$$

Thus our solution, $\boxed{x = -30}$ is correct.

7. $-\frac{2}{5}x + \frac{1}{3} = \frac{4}{15}x$

Solution:

$$\begin{aligned}
-\frac{2}{5}x + \frac{1}{3} &= \frac{4}{15}x & & \text{add } \frac{2}{5}x \text{ to both sides} \\
\frac{1}{3} &= \frac{2}{3}x & & \text{divide both sides by } \frac{2}{3} \\
\frac{1}{2} &= x
\end{aligned}$$

The computation for each step are as follows. To add $\frac{2}{5}x$ to the right-hand side:

$$\frac{4}{15}x + \frac{2}{5}x = \left(\frac{4}{15} + \frac{2}{5}\right)x = \left(\frac{4}{15} + \frac{6}{15}\right)x = \frac{10}{15}x = \frac{2}{3}x$$

To divide by $\frac{2}{3}$ is to multiply by its reciprocal:

$$\frac{1}{3} \div \frac{2}{3} = \frac{1}{3} \cdot \frac{3}{2} = \frac{1}{2}$$

We check: if $x = \frac{1}{2}$, then

$$\begin{aligned}
\text{LHS} &= -\frac{2}{5}\left(\frac{1}{2}\right) + \frac{1}{3} = -\frac{1}{5} + \frac{1}{3} = \frac{-3}{15} + \frac{5}{15} = \frac{-3+5}{15} = \frac{2}{15} \\
\text{RHS} &= \frac{4}{15}\left(\frac{1}{2}\right) = \frac{4}{30} = \frac{2}{15}
\end{aligned}$$

Thus our solution, $\boxed{x = \frac{1}{2}}$ is correct.

8. $4 - x = 3(x - 7)$

   Solution: We first apply the law of distributivity to simplify the right-hand side.

$$
\begin{array}{rcl l}
4 - x & = & 3(x - 7) & \text{distribute } 3 \\
4 - x & = & 3x - 21 & \text{add } x \text{ to both sides} \\
4 & = & 4x - 21 & \text{add } 21 \text{ to both sides} \\
25 & = & 4x & \text{divide both sides by } 4 \\
\dfrac{25}{4} & = & x &
\end{array}
$$

We check. If $x = \dfrac{25}{4}$, then

$$
\begin{array}{rcl}
\text{LHS} & = & 4 - x = 4 - \dfrac{25}{4} = \dfrac{4}{1} - \dfrac{25}{4} = \dfrac{16}{4} - \dfrac{25}{4} = \dfrac{16 - 25}{4} = \dfrac{-9}{4} = -\dfrac{9}{4} \\[2mm]
\text{RHS} & = & 3(x - 7) = 3\left(\dfrac{25}{4} - 7\right) = 3\left(\dfrac{25}{4} - \dfrac{7}{1}\right) = 3\left(\dfrac{25}{4} - \dfrac{28}{4}\right) = 3\left(\dfrac{25 - 28}{4}\right) \\[2mm]
& = & 3\left(\dfrac{-3}{4}\right) = \dfrac{3}{1} \cdot \dfrac{-3}{4} = \dfrac{-9}{4} = -\dfrac{9}{4}
\end{array}
$$

Thus our solution, $\boxed{x = \dfrac{25}{4}}$ is correct.

9. $7(j - 5) + 9 = 2(-2j + 5) + 5j$

   Solution:

$$
\begin{array}{rcl l}
7(j - 5) + 9 & = & 2(-2j + 5) + 5j & \text{distribute on both sides} \\
7j - 35 + 9 & = & -4j + 10 + 5j & \text{combine like terms} \\
7j - 26 & = & j + 10 & \text{subtract } j \\
6j - 26 & = & 10 & \text{add 26} \\
6j & = & 36 & \text{divide by 6} \\
j & = & 6 &
\end{array}
$$

We check: if $j = 6$, then

$$
\begin{array}{rcl}
\text{LHS} & = & 7(6 - 5) + 9 = 7 \cdot 1 + 9 = 7 + 9 = 16 \\
\text{RHS} & = & 2(-2 \cdot 6 + 5) + 5 \cdot 6 = 2(-12 + 5) + 30 = 2(-7) + 30 = -14 + 30 = 16
\end{array}
$$

Thus our solution, $\boxed{j = 6}$ is correct.

10. $3(x - 5) - 5(x - 1) = -2x + 1$

    Solution:

$$
\begin{array}{rcl l}
3(x - 5) - 5(x - 1) & = & -2x + 1 & \text{multiply out parentheses} \\
3x - 15 - 5x + 5 & = & -2x + 1 & \text{combine like terms} \\
-2x - 10 & = & -2x + 1 & \text{add } 2x \\
-10 & = & 1 &
\end{array}
$$