We will study solving linear inequalities. Let us first recall a few definitions.

> **Definition**: An **inequality** is a statement in which two expressions (algebraic or numeric) are connected with one of $<$, $\leq$, $>$, or $\geq$. A **solution** of an inequality is a number that, when substituted into the variable in the inequality, makes the statement of inequality true. To **solve an inequality** is to find *all* solutions of it. The set of all solutions is also called the solution set.

Example 1. Consider the inequality $-3x+8<-2(x-1)+5$. In case of each of the numbers given, determine whether it is a solution of the inequality or not.

a) -3 b) 4 c) 1 d) 8

Solution: a) We substitute the given number into the variable and check whether the inequality statement is true.

$\text{LHS} = -3(-3)+8=17$ and $\text{RHS} = -2(-3-1)+5=-2(-4)+5=8+5=13$

The statement $17<13$ is false. Thus -3 is $\boxed{\text{not a solution}}$ of the inequality.

b) We substitute 4 into the variable and check whether the inequality statement is true.

$\text{LHS} = -3\cdot 4+8=-12+8=-4$ and $\text{RHS} = -2(4-1)+5=-2\cdot 3+5=-6+5=-1$

The statement $-4<-1$ is true. Thus 4 is $\boxed{\text{a solution}}$ of the inequality.

c) We substitute 1 into the variable and check whether the inequality statement is true.

$\text{LHS} = -3\cdot 1+8=-3+8=5$ and $\text{RHS} = -2(1-1)+5=-2\cdot 0+5=0+5=5$

The statement $5<5$ is false. Thus 1 is $\boxed{\text{not a solution}}$ of the inequality.

d) We substitute 8 into the variable and check whether the inequality statement is true.

$\text{LHS} = -3\cdot 8+8=-24+8=-16$ and $\text{RHS} = -2(8-1)+5=-2\cdot 7+5=-14+5=-9$

The statement $-16<-9$ is true. Thus 8 is $\boxed{\text{a solution}}$ of the inequality.

Inequalities have many, many solutions. To express these much larger solution sets, we developed interval notation. The steps of solving a liner inequality are almost identical to those of solving linear equations. However, there is a very important difference. Consider the true inequality $3\leq 7$. If we add or subtract the same number from both sides, the inequality will remain true. The result is the same if we multiply both sides by a positive number. But if we multiply both sides of $3\leq 7$ by -2, we get $-6\leq -14$, which is false.

> **When multiplying or dividing both sides of an inequality by a negative number, we must reverse the inequality sign.**

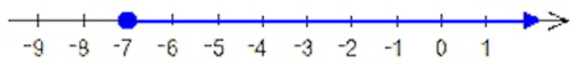
Example 2. Solve the inequality $\dfrac{-3x+1}{2}\leq 11$

Solution: The steps are identical to those of solving equations, except for when multiplying or dividing by a negative number.

$$\begin{aligned}
\frac{-3x+1}{2} &\leq 11 && \text{multiply by } 2\\
-3x+1 &\leq 22 && \text{subtract } 1\\
-3x &\leq 21 && \text{divide by } -3 \implies \text{MUST reverse inequality sign}\\
x &\geq -7
\end{aligned}$$

Notice that we reversed the inequality sign when we divided by -3. Our solution set is the set of all numbers greater than or equal to -7.

We can present this set as an interval: $[-7, \infty)$. We can also depict the solution set on the number line:



Although we will not be asked to do so, we can check inequalities too. If we randomly pick a number inside our solution set and substitute it into the inequality, the inequality statement must be true. If we randomly pick a number outside our solution set and substitute it into the inequality, the inequality statement must be false. Finally, if substitute the boundary point (the one that separates the solutions from the non-solutions) the two sides should be equal. For an easy to substitute number inside our solution set, we choose $x = 0$. Substituting it into the inequality, we get $\frac{1}{2} \leq 11$, which is true. For a number outside of our solution set, we choose $x = -10$. This value results in $\frac{31}{2} \leq 11$, which is false. Finally, setting $x = -7$, we get the two sides equal.

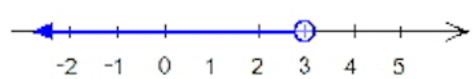
Example 3. Solve the given inequality. Present the solution set using interval notation and plot it on a number line.

$$\frac{5x+1}{4} - \frac{2x-1}{5} > 2x - 3$$

Solution: We will bring both sides to the common denominator, and then clear all denominators by multiplying by that number. In fact, we can speed up the process by just multiplying both sides by the common denominator. In this case, that common denominator is 20.

$$\begin{array}{rcll}
\frac{5x+1}{4} - \frac{2x-1}{5} & > & 2x-3 & \text{multiply by } 20 \\
5(5x+1) - 4(2x-1) & > & 20(2x-3) & \text{expand products} \\
25x + 5 - 8x + 4 & > & 40x - 60 & \text{combine like terms} \\
17x + 9 & > & 40x - 60 & \text{subtract } 17x \\
9 & > & 23x - 60 & \text{add } 60 \\
69 & > & 23x & \text{divide by } 23 \\
3 & > & x &
\end{array}$$

Our solution is $(-\infty, 3)$. We depict the solution set on a number line:



In case it is not clear how we got from the first line to the second line, here is the detailed computation:

$$20\left(\frac{5x+1}{4} - \frac{2x-1}{5}\right) = 20 \cdot \frac{5x+1}{4} - 20 \cdot \frac{2x-1}{5} = \frac{20(5x+1)}{4} - \frac{20(2x-1)}{5} = 5(5x+1) - 4(2x-1)$$

We recommend however to perform these steps on the margin or mentally.

Solve each of the following inequalities. Graph the solution set.

1. $-7 > -5x + 3$
2. $3(x-2) \leq 2x + 1$
3. $5(4x-1) - (x-3) \geq -x - 2$
4. $\dfrac{m+4}{2} - \dfrac{4m+3}{5} > 2$

Practice Problems

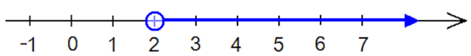
Solve each of the following inequalities. Graph the solution set.

1. $x - 17 > -4x + 3$
2. $-3x + 5 \leq 12$
3. $5y + 3 < y - 7$
4. $-2x - (3x-1) \geq 2(5-3x)$
5. $\dfrac{2}{3}x - 1 \geq x$
6. $5 - (3a-2) < -2$
7. $5x - 2 > 3(x-1) - 4x + 1$
8. $-3(x-2) \leq -2x + 5$
9. $3x - 2(x-1) < -2x - 1$
10. $-w + 13 \geq 2w + 1$
11. $2x + 5 > \dfrac{3x-1}{2} - \dfrac{2x+1}{3}$
12. $5(x-1) - 3(x+1) \geq 3x - 8$
13. $3(x-4) + 5(x+8) \leq 2(x-1)$
14. $2x + 6 > \dfrac{3x-1}{5} - \dfrac{7-x}{3}$
15. $-\dfrac{2}{5}(x+1) + \dfrac{1}{2}(x-4) \geq \dfrac{3}{10}x$
16. $\dfrac{3x-1}{4} + \dfrac{8-4x}{3} \leq -3 - x$
17. $\dfrac{x-2}{5} - \dfrac{x}{2} < x - 16$
18. $\dfrac{2x+1}{3} + 2 \geq x + \dfrac{3-x}{2}$

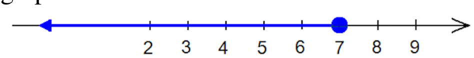
Answers

Sample Problems

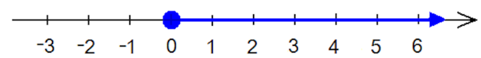
1. set-builder notation: $\{x|x > 2\}$
   interval notation: $(2, \infty)$
   graph:



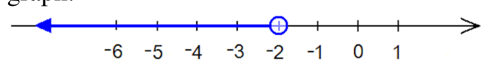
2. set-builder notation: $\{x|x \leq 7\}$
   interval notation: $(-\infty, 7]$
   graph:



3. set-builder notation: $\{x|x \geq 0\}$
   interval notation: $[0, \infty)$
   graph:

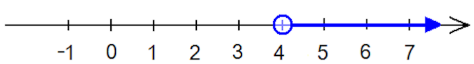


4. set-builder notation: $\{x|x < -2\}$
   interval notation: $(-\infty, -2)$
   graph:

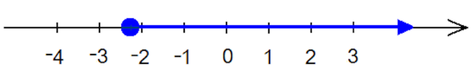


Practice Problems

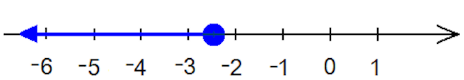
1. set-builder notation: $\{x|x > 4\}$
   interval notation: $(4, \infty)$
   graph:



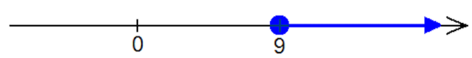
2. set-builder notation: $\left\{x|x \geq -\frac{7}{3}\right\}$
   interval notation: $\left[-\frac{7}{3}, \infty\right)$
   graph:



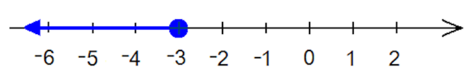
3. set-builder notation: $\left\{x|x < -\frac{5}{2}\right\}$
   interval notation: $\left(-\infty, -\frac{5}{2}\right)$
   graph:



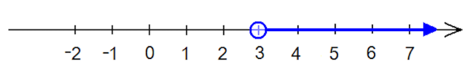
4. set-builder notation: $\{x|x \geq 9\}$
   interval notation: $[9, \infty)$
   graph:



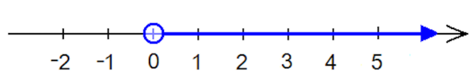
5. set-builder notation: $\{x|x \leq -3\}$
   interval notation: $(-\infty, -3]$
   graph:



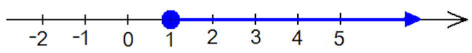
6. set-builder notation: $\{x|x > 3\}$
   interval notation: $(3, \infty)$
   graph:



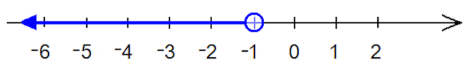
7. set-builder notation: $\{x|x > 0\}$
   interval notation: $(0, \infty)$
   graph:



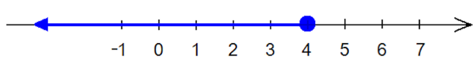
8. set-builder notation: $\{x|x \geq 1\}$
interval notation: $[1, \infty)$
graph:



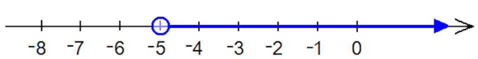
9. set-builder notation: $\{x|x < -1\}$
interval notation: $(-\infty, -1)$
graph:



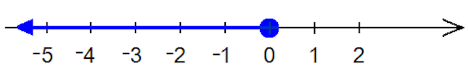
10. set-builder notation: $\{x|x \leq 4\}$
interval notation: $(-\infty, 4]$
graph:



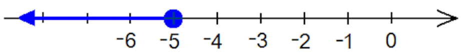
11. set-builder notation: $\{x|x > -5\}$
interval notation: $(-5, \infty)$
graph:



12. set-builder notation: $\{x|x \leq 0\}$
interval notation: $(-\infty, 0]$
graph:

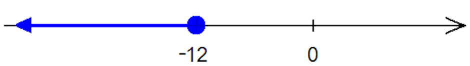


13. set-builder notation: $\{x|x \leq -5\}$
interval notation: $(-\infty, -5]$
graph:

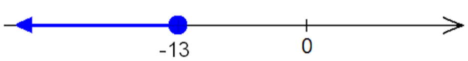


14. set-builder notation: $\{x|x > -8\}$
interval notation: $(-8, \infty)$
graph:

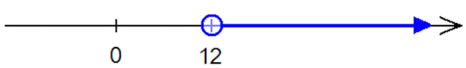
15. set-builder notation: $\{x|x \leq -12\}$
interval notation: $(-\infty, -12]$
graph:



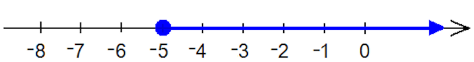
16. set-builder notation: $\{x|x \leq -13\}$
interval notation: $(-\infty, -13]$
graph:



17. set-builder notation: $\{x|x > 12\}$
interval notation: $(12, \infty)$
graph:



18. set-builder notation: $\{x|x \geq -5\}$
interval notation: $[-5, \infty)$
graph:



Sample Problems - Solutions

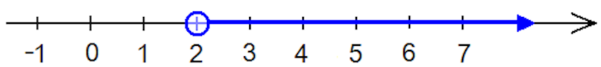
1. $-7 > -5x + 3$

Solution: Solving linear inequalities requires almost the same techniques as solving linear equations. There is only one difference: when multiplying or dividing an inequality by a negative number, the inequality sign must be reversed.

$$\begin{array}{rcll} -7 & > & -5x+3 & \text{subtract } 3 \\ -10 & > & -5x & \text{divide by } -5 \\ 2 & < & x & \end{array}$$

When we divided both sides by -5, we reversed the inequality sign. The final answer is all real numbers greater than 2. This set of numbers can be presented in numerous ways:

1) set-builder notation: $\{x | x > 2\}$
2) interval notation: $(2, \infty)$
3) graphing the solution set on the number line:



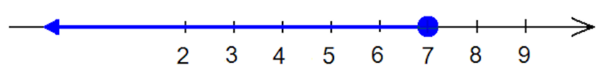
2. $3(x-2) \leq 2x+1$

Solution:

$$\begin{array}{rcll} 3(x-2) & \leq & 2x+1 & \text{distribute} \\ 3x-6 & \leq & 2x+1 & \text{subtract } 2x \\ x-6 & \leq & 1 & \text{add } 6 \\ x & \leq & 7 & \end{array}$$

The final answer is all real numbers less than or equal to 7. This set of numbers can be presented in numerous ways:

1) set-builder notation: $\{x | x \leq 7\}$
2) interval notation: $(-\infty, 7]$
3) graphing the solution set on the number line:

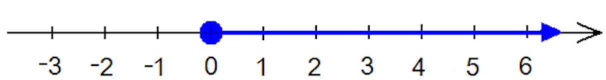


3. $5(4x-1) - (x-3) \geq -x-2$

Solution:

$$\begin{array}{rcll} 5(4x-1)-(x-3) & \geq & -x-2 & \text{distribute} \\ 20x-5-x+3 & \geq & -x-2 & \text{combine like terms} \\ 19x-2 & \geq & -x-2 & \text{add } 2 \\ 19x & \geq & -x & \text{add } x \\ 20x & \geq & 0 & \text{divide by } 20 \\ x & \geq & 0 & \end{array}$$

The final answer is all real numbers greater than or equal to 0. This set of numbers can be presented in numerous ways: in set-builder notation: $\{x | x \geq 0\}$, in interval notation: $[0, \infty)$, or by graphing the solution set on the number line:



4. $\dfrac{m+4}{2}-\dfrac{4m+3}{5}>2$

Solution:

$$\begin{array}{rcll}
\dfrac{m+4}{2}-\dfrac{4m+3}{5} & > & 2 & \text{make everything a fraction} \\
\dfrac{m+4}{2}-\dfrac{4m+3}{5} & > & \dfrac{2}{1} & \text{bring to common denominator} \\
\dfrac{5(m+4)}{10}-\dfrac{2(4m+3)}{10} & > & \dfrac{20}{10} & \text{multiply by 15} \\
5(m+4)-2(4m+3) & > & 20 & \text{distribute} \\
5m+20-8m-6 & > & 20 & \text{combine like terms} \\
-3m+14 & > & 20 & \text{subtract 14} \\
-3m & > & 6 & \text{divide by } -3 \\
m & < & -2 &
\end{array}$$

When we divided both sides by -3, we reversed the inequality sign. The final answer is all real numbers less than -2. This set of numbers can be presented in numerous ways: in set-builder notation: $\{x|x<-2\}$, in interval notation: $(-\infty,-2)$, or by graphing the solution set on the number line: