

Definition: Suppose that N and m are any two integers. If there exists an integer k such that $N = mk$, then we say that m is a **factor** or **divisor** of N . We also say that N is a **multiple** of m or that N is **divisible** by m . Notation: $m|N$

For example, 3 is a factor of 15 because there exists another integer (namely 5) so that $3 \cdot 5 = 15$.

Notation: $3|15$.

Example 1. Label each of the following statements as true or false.

- a) 2 is a factor of 10 b) 3 is divisible by 3 c) 14 is a factor of 7 d) 0 is a multiple of 5
 e) every integer is divisible by 1 f) every integer n is divisible by n

Solution: a) $10 = 2 \cdot 5$ and so 2 is a factor of 10. This statement is true.

b) $3 = 3 \cdot 1$ and so 3 is divisible by 3. This statement is true.

c) $14 = 7 \cdot 2$ and so 14 is a multiple of 7, not a factor. Can we find an integer k so that $7 = 14 \cdot k$? This is not possible. $k = \frac{1}{2}$ would work, but $\frac{1}{2}$ is not an integer. This statement is false.

d) Since $0 = 5 \cdot 0$, it is indeed true that 0 is a multiple of 5. This statement is true.

e) For any integer n , $n = n \cdot 1$ and so every integer n is divisible by 1. This statement is true.

f) For any integer n , $n = 1 \cdot n$ and so every integer n is divisible by n . This statement is true.



Discussion: What do you think about the argument shown below?

3 is a divisor of 21 because there exists another integer, namely 7 so that $21 = 3 \cdot 7$. As we established that 3 is a divisor of 21, we also found that 7 is also a divisor of 21. In other words, divisors always come in pairs. For example, 28 has six divisors that we found in three pairs: 1 with 28, 2 with 14, and 4 with 7. Consequently, every positive integer has an even number of positive divisors.

Definition: An integer is a **prime number** if it has exactly two divisors: 1 and itself.

For example, 37 is a prime number. Prime numbers are a fascinating study within mathematics. Let us first recall the definition. Given a number n , we can find all of its divisors. For example, $n = 20$ has six divisors: 1, 2, 4, 5, 10, and 20. Prime numbers have a very short list of divisors: only the trivial divisors, 1 and the number itself. For example, 7 is a prime number. 6 is not a prime number since it has divisors other than 1 and 6.

It is important to notice that 1 is not a prime number. The first few prime numbers are: 2, 3, 5, 7, 11, 13, 17, 19, ... With respect to multiplication, prime numbers are the basic building blocks of numbers.

Theorem: (*Fundamental Theorem of Arithmetic*) Every integer greater than 1 can be written as a product of prime numbers, and this decomposition is unique up to order of factors.

For example, $20 = 2 \cdot 2 \cdot 5$. According to the fundamental theorem of arithmetic, there is no other way to write 20 as a product of prime numbers. We usually use exponential notation: $20 = 2^2 \cdot 5$.

Example 2. Find the prime factorization of 180.

Solution: We start with the smallest prime number, 2 and ask: is 180 divisible by 2? If the answer is yes, we write down and divide 180 by 2.

$$\begin{array}{r|l} 180 & 2 \\ 90 & \end{array}$$

Now we ask: is 90 divisible by 2? The answer is yes. So we write 2 next to 90 and divide by 2.

$$\begin{array}{r|l} 180 & 2 \\ 90 & 2 \\ 45 & \end{array}$$

Now we ask: is 45 divisible by 2? The answer is no. So, we are done with the prime factor 2 and move on to the next prime number, 3. Since 45 is divisible by 3, we write it next to 45 and divide by 3.

$$\begin{array}{r|l} 180 & 2 \\ 90 & 2 \\ 45 & 3 \\ 15 & \end{array}$$

Once we wrote 1, we are done. The prime factorization of 180 is therefore

$$180 = 2 \cdot 2 \cdot 3 \cdot 3 \cdot 5 \text{ or } 180 = 2^2 \cdot 3^2 \cdot 5$$

Now we ask: is 15 divisible by 3? The answer is yes. So we write 3 next to 15 and divide by 3.

$$\begin{array}{r|l} 180 & 2 \\ 90 & 2 \\ 45 & 3 \\ 15 & 3 \\ 5 & \end{array}$$

The only prime divisor of 5 is 5 itself. We write 5 next to 5 and then divide.

$$\begin{array}{r|l} 180 & 2 \\ 90 & 2 \\ 45 & 3 \\ 15 & 3 \\ 5 & 5 \\ 1 & \end{array}$$

Example 3. a) Find the prime factorization of 300.

b) Find the prime factorization of 2250. [Solution - Youtube link](#)

Solution: We start with the first prime number, 2. Is our number, 300 divisible by 2? If yes, we divide 300 by 2. Since $300 = 2 \cdot 150$, we now have one prime factor, 2 and we must find the prime factorization of 150.

We ask next: is the number 150 divisible by 2? If yes, we divide 150 by 2. So now $300 = 2 \cdot 2 \cdot 75$ and we are looking for the prime factorization of 75.

We ask next: is the number 75 divisible by 2? This time, the answer is no. We have exhausted the prime factor 2. So we roll up to 3 and ask: is 75 divisible by 3? If yes, we divide 75 by 3. Since $75 \div 3 = 25$, so now $300 = 2 \cdot 2 \cdot 3 \cdot 25$ and we are looking for the prime factorization of 25. Although we know the final answer now, we continue the process. Every time we find a factor, we write it down and divide. The quotient is written down under the pair in the first column. Once that column reaches 1, the second column is the prime factorization of our number. Thus $300 = 2 \cdot 2 \cdot 3 \cdot 5 \cdot 5 = 2^2 \cdot 3 \cdot 5^2$

$$\begin{array}{r|l} 300 & 2 \\ 150 & 2 \\ 75 & 3 \\ 25 & 5 \\ 5 & 5 \\ 1 & \end{array}$$

Prime Numbers

Suppose we are given a number N and asked to determine if the number is a prime number or not. In theory, this is an easy question to answer. We just take the numbers, $1, 2, 3, \dots, N$ and check each whether they are a divisor of N or not. If we checked all these numbers and find that the only divisors are the trivial divisors 1 and N , the number is a prime number. If we find a non-trivial divisor, the number is not a prime.

One interesting thing we can notice is that the level of difficulty depends on the answer. For example, we can use the divisibility rule for 3 to find that the number 10 300 800 050 703 is divisible by 3 (A number is divisible by 3 if and only if the sum of its digits is divisible by 3). So 3 is a non-trivial divisor and so 10 300 800 050 703 is not a prime number. Consider now the number 10 300 800 050 719. If this is a prime number, that means that as we check all numbers from 2 to 10 300 800 050 711, none of them would be a divisor. So we would have to keep going and check millions and millions of numbers. So whether 10 300 800 050 703 is a prime or not is an easy question, while the same question for 10 300 800 050 719 (assuming it is a prime) is a very difficult question.

While in theory we just need to check all numbers between 1 and the given number, this can be very difficult in practice. Even in the age of computers, if a number is great enough, in practice it is nearly impossible to answer this question. *Cryptography* is a branch of mathematics that studies secret codes. To this day, the safest, most difficult to break codes are based on very large prime numbers.

If we are given a number N and we need to determine whether it is a prime number or not, there are two important ways we can reduce our work.

Theorem: Given an integer $N \geq 2$. If N is not a prime number, then the smallest non-trivial divisor is a prime number.

This means that *if we check the numbers, $2, 3, 4, \dots, N$ in this order*, then we can skip all composite numbers and check only the prime numbers.

Suppose we are given a number N and we need to see if it is a prime number. Let us also suppose that we are looking for divisors starting at 2, then 3, and so on, in an increasing order.

Suppose that our number N is divisible by 77. Then there exists an integer k , such that $N = 77k$. We can express 77 as $7 \cdot 11$. Then

$$N = 77k = 7 \cdot 11 \cdot k = 7(11k)$$

Therefore, if N is divisible by 77, then it is also divisible by 7.

We are not asked to list all divisors, we only want to know if our number is a prime number. Therefore, we can stop searching the moment we find just one non-trivial divisor. If we are checking for divisors in an increasing order, reaching 77 means that all previous numbers, including 7 and 11 failed to be divisors. So, if we didn't find a divisor until 77, we can skip 77 because divisibility by 77 means that N is also divisible by 7 - but then we would have stopped right there.

So, when checking for divisors of N among $2, 3, \dots, N$, in an increasing order, we can skip all composite numbers and check for prime divisors only.

Theorem: Given an integer $N \geq 2$. Suppose further than x is an integer such that $x^2 > N$. Then if we did not find a divisor between 2 and x , the number is a prime number.

Recall that divisors come in pairs. Let us consider the number 36. The pair of divisors are $1 \cdot 36$, $2 \cdot 18$, $3 \cdot 12$, $4 \cdot 9$ and $6 \cdot 6$. In every pair of divisors, (except 6 with itself), the smaller one is less than 6, the greater one is greater than 6.

Suppose a and b are integers, both greater than 6. Then ab is greater than 36. So if we have a pair of divisors and one is greater than 6, then the other one must be less than 6.

Suppose we would like to determine if 307 is a prime number or not. If we enter $\sqrt{307}$ into the calculator, we get a decimal $\sqrt{307} = 17.52141\dots$. This means that 307 is between 17^2 and 18^2 . Indeed, $17^2 = 289$ and $18^2 = 324$. Let us round the number 17.52141... up, we get 18. Since $18^2 = 324$, (greater than 307), in any pair of divisors, one of them is must be 18 or less. This is because if $a, b > 18$, then $ab > 324$.

Suppose we would like to decide whether 307 is a prime number. If we don't find a non-trivial divisor until 18, then our number, 307, is a prime number.

Example 4. For each of the numbers given, determine whether it is a prime number or not.

a) 307 b) 57 c) 59 [Solution - Youtube link](#)

Solution: Given the two facts established above, 307 is a prime number if none of the prime numbers between 2 and 18 is a divisor of 307. These primes are: 2, 3, 5, 7, 11, 13, and 17. We can use the calculator or long division. For example, when we check 7, and enter $307 \div 7$, the result is 43.8571... This is not an integer, therefore 7 is not a divisor of 307. When we check all of 2, 3, 5, 7, 11, 13, and 17, none of them is a divisor of 307. Therefore, 307 is a prime number.

Example 5. Determine whether 401 is a prime number or not.

Solution: We enter $\sqrt{401}$ into the calculator. The result is $\sqrt{401} = 20.02498\dots$. This means that 21 is the smallest number such that its square is greater than 400. Therefore, 401 is a prime number if none of 2, 3, 5, 7, 11, 13, 17, and 19 is a divisor of 401. We check all these numbers and find that none of them is a divisor. Therefore, 401 is a prime number.

Example 6. For each of the numbers given, determine whether it is a prime number or not.

a) 1037 b) 1139 [Solution - Youtube link](#)

Solution: We enter $\sqrt{1037}$ into the calculator. The result is $\sqrt{1037} = 32.202484\dots$. This means that 33 is the smallest number such that its square is greater than 1037. Therefore, 1037 is a prime number if none of 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, and 31 is a divisor of 1037. We check all these numbers and find $1037 \div 17 = 61$. Therefore, 1037 is not a prime number since it is divisible by 17, so it has more than two divisors.

Example 7. Find all prime numbers from 21, 37, 77, 89, 93. [Solution - Youtube link](#)



Practice Problems

1. Consider the numbers 64, 75, 80, 128, 270. From this list, find all numbers that are divisible by
a) 5 b) 3 c) 4.
2. List all factors of 48.
3. Which of 53, 73, 91, 101, 139 is NOT a prime number?
4. Find the prime factorization for each of the following numbers.
a) 600 b) 5500 c) 2016 d) 2015
5. Given the numbers, determine about each of them whether it is a prime number or not.
a) 2017 c) 2021 e) 2027 g) 4003
b) 2019 d) 2023 f) 4001 h) 4009
6. What is the smallest prime number greater than
a) 100 b) 1000?



Enrichment

1. Magic: think of a three digit number. Enter a six-digit number into your calculator by repeating your three-digit number twice. For example, if you thought of the three-digit number 275, then enter 275275 into your calculator. Done? No matter what number you used to start, the number in your calculator is divisible by 7. Divide by 7. The number in your calculator now is still divisible by 11. Divide it by 11. The number in your calculator is still divisible by 13. Divide it by 13. What do you see? Can you explain it?
2. Two mathematicians are having a conversation. Mathematician A asks B about his kids. B answers: "I have three children, the product of their ages is 36." A says: "I still don't know the ages of your children." Then B tells A the sum of his three kids' ages. A answers: "I still don't know how old they are. Then B adds: "The youngest one has red hair." Now A knows the ages of all three children. Do you?
3. A king has his birthday. So he decides to let go some of his prisoners. He actually has 100 prisoners at the moment. They are each in a separate cell, numbered from 1 to 100. Well, he is a high tech king. He can close or open any prison door by a single click on the cell's number on his royal laptop. When he clicks at a locked door, it opens. When he clicks at an open door, it locks. At the beginning, every door is locked. First the king clicks on every number from 1 to 100 (therefore opening every door). Then he clicks on every second number from 1 to 100, (i.e. 2, 4, 6, 8, 10,...). Then he clicks on every third number. And so on. Finally, he only clicks on the number 100. Then he orders that the prisoners who find their door open may go free. Who gets to go and who has to stay?



Answers

Practice Problems

1. 1, 2, 3, 4, 6, 8, 12, 16, 24, 48 2. 91 3. a) 80, 75, 270 b) 75, 270 c) 128, 80, 64
4. a) $600 = 2^3 \cdot 3 \cdot 5^2$ b) $5500 = 2^2 \cdot 5^3 \cdot 11$ c) $2016 = 2^5 \cdot 3^2 \cdot 7$ d) $2015 = 5 \cdot 13 \cdot 31$
5. a) yes b) no, $2019 = 3 \cdot 673$ c) no, $2021 = 43 \cdot 47$ d) no, $2023 = 7 \cdot 289$ e) yes f) yes g) yes h) no, $4009 = 19 \cdot 211$
6. a) 101 b) 1009, because $1001 = 7 \cdot 143$, $1003 = 17 \cdot 59$ and $1007 = 19 \cdot 53$ (1002, 1004, 1006, and 1008 are divisible by 2 and 1005 by 5)