

Definition: The **set of all natural numbers**, denoted by $\mathbb{N}$, is the infinite set

$$\mathbb{N} = \{1, 2, 3, 4, \dots\}$$

If we add two natural numbers, the sum is also a natural number. In other words, if x and y are natural numbers, then the sum $x + y$ is also a natural number. When this is true, we say that the set of all natural numbers is **closed under addition**. On the other hand, the set of all natural numbers is not closed under subtraction: while $10 - 3$ is a natural number, $3 - 10$ is not.

Theorem: *The set of all natural numbers is closed under addition and multiplication, but not under subtraction and division.*

Definition: The **set of all integers**, denoted by $\mathbb{Z}$, is the infinite set

$$\mathbb{Z} = \{0, 1, -1, 2, -2, 3, -3, \dots\}$$

Notice that the set of all integers contains all natural numbers. When this happens, we say that the set of all natural numbers is a subset of the set of all integers. Notation: $\mathbb{N} \subseteq \mathbb{Z}$.

Theorem: *The set of all integers is closed under addition, multiplication, and subtraction, but not under division.*

Definition: A number is **rational** if it can be written as a quotient of two integers.

For example, $\frac{3}{8}$ is a rational number because both 3 and 8 are integers and so $\frac{3}{8}$ is a quotient of two integers.

Definition: The **set of all rational numbers**, denoted by $\mathbb{Q}$, is the infinite set

$$\mathbb{Q} = \left\{ \frac{a}{b} : a \text{ and } b \text{ are integers, } b \neq 0 \right\}$$

Notice that the set of all rational numbers entirely contains the set of all integers, i.e. $\mathbb{Z} \subseteq \mathbb{Q}$. In fact, the three sets are such that

$$\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q}$$

For example, the number -5 is an integer and also a rational number, because we can write it as a quotient $\frac{-5}{1}$ where both -5 and 1 are integers. The number zero is also a rational number because it can be written as $\frac{0}{3}$.

Theorem: *The set of all rational numbers is closed under addition, multiplication, subtraction, and division.*

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