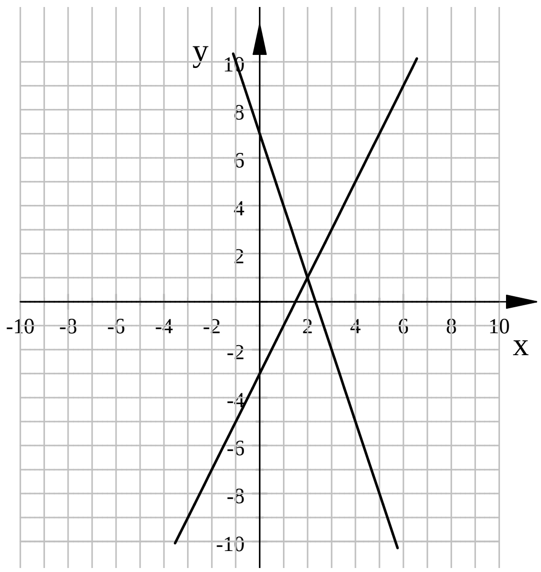
As we are progressing in algebra, we have learned how to solve linear equations and inequalities. The solution set of a linear equation was usually very simple: a single number. The set of all solutions of an inequality is much more complicated. We can no longer just list all elements in the solution set, and so we needed to develop new notation: interval notation.

Straight lines are even more complicated solution sets. They are solution set of a linear equation in two variables. Consider, for example, the graph of the equation $y = 2x - 3$. Every point on the graph of this line have coordinates that form a solution to the equation $y = 2x - 3$. For example, points such as $(7, 11)$ and $(10, 17)$, and $(0, -3)$ are all in this solution set, because $11 = 2 \cdot 7 - 3$, $17 = 2 \cdot 10 - 3$, and $-3 = 2 \cdot 0 - 3$. As a matter of fact, x can be any real number and then there is a unque real number that will work for y. This set is the set of points $P(x, 2x - 3)$. The most meaningful representation of this set might just be its graph.



Consider now two equations in x and y. Our example will be $y = 2x - 3$ and $y = -3x + 7$. If we graphed the two equations in the same coordinate system, we would see two straight lines. If the lines are not parallel, they intersect each other in a point. In this case, this point appears to be $(2, 1)$. Can we use algebraic methods to see if the point $(2, 1)$ is the intersection point? The intersection point is the only point that is contained in both lines. Indeed, $1 = 2 \cdot 2 - 3$, and so $x = 2, y = 1$ is a solution of $y = 2x - 3$. Also, $1 = -3 \cdot 2 + 7$ and so $x = 2$, $y = 1$ is also a solution of $y = -3x + 7$. Therefore, the point $(2, 1)$ is on both lines and must be the intersection point.

We also say that the ordered pair $(2, 1)$ is a symultaneous solution of both equations.

Definition: Two equations in x and y form a **system of equations**. The **solution**(s) of the system are the point(s) whose coordinates form a solution of both equations. To **solve** a system means to find all solutions of it.

In our example above, $(2, 1)$ is the only solution of the system $\begin{cases} y = 2x - 3 \\ y = -3x + 7 \end{cases}$.

In this case, the intersection was a point whose both coordiantes happen to be integers. Such points are called lattice points. In case we are less lucky, we will need more precise tools for solving than graphing the two equations in the same cordinate system. There are several algebraic methods to solve a system of linear equations. Here we will explore a technique called substitution.

Solving Linear Systems using Substitution

The basic idea of this method is to absorb the information of one equation and to substitute that into the other equation, thereby reducing the number of unknowns to one.

Example 1. Solve the given system of linear equations using substitution. $\begin{cases} 2x - y = -19 \\ -x + 3y = 12 \end{cases}$

Solution: We first inspect the two equations and look for coefficients such as 1 or -1. In this case, the coefficient of y is -1 in the first equation. We solve for y in this equation. We can't solve for y and obtain a number, we can only solve for it in terms of x.

$$\begin{aligned} 2x - y &= -19 && \text{add } y \\ 2x &= y - 19 && \text{add } 19 \\ 2x + 19 &= y \end{aligned}$$

The information from the first equation can be expressed as $\boxed{y = 2x + 19}$. This is going to be what we substitute into the other equation by substituting $2x + 19$ into y. This way, the equation $x + 3y = 12$ will become $-x + 3(2x + 19) = 12$. This is now an equation in only one variable for which we can solve.

$$\begin{aligned} -x + 3(2x + 19) &= 12 \\ -x + 6x + 57 &= 12 \\ 5x + 57 &= 12 && \text{subtract } 57 \\ 5x &= -45 && \text{divide by } 5 \\ x &= -9 \end{aligned}$$

Now that we know the value of x, we return to what we used for substitution and get the value of the other unknown.

$$y = 2x + 19 = 2(-9) + 19 = -18 + 19 = 1$$

Therefore, the solution of this system is $x = -9$ and $y = 1$, or, in short, $(-9, 1)$. We check: the solution of a system is a simultaneous solution of both equations.

Checking $2x - y = -19$

$$\begin{aligned} \text{LHS} &= 2(-9) - 1 = -18 - 1 = -19 \\ \text{RHS} &= -19 \checkmark \end{aligned}$$

Checking $-x + 3y = 12$

$$\begin{aligned} \text{LHS} &= -(-9) + 3 \cdot 1 = 9 + 3 = 12 \\ \text{RHS} &= 12 \checkmark \end{aligned}$$

Therefore, our solution, $\boxed{(-9, 1)}$ is correct.

Of course, not all linear systems contain easy coefficients such as 1 or -1.

Example 2. Solve the given system of linear equations. $\begin{cases} 3x - 5y = 11 \\ 2x + 3y = 20 \end{cases}$

Solution: We will solve for x in the second equation and substitute the information into the first equation. First, we solve for x in $2x + 3y = 20$.

$$\begin{aligned} 2x + 3y &= 20 && \text{subtract } 3y \\ 2x &= -3y + 20 && \text{divide by } 2 \\ x &= \frac{-3y + 20}{2} \end{aligned}$$

This is the information we will substitute into the first equation. $3x - 5y = 11$ will become $3\left(\dfrac{-3y+20}{2}\right) - 5y = 11$. We solve this equation for y.

$$\begin{aligned} 3\left(\frac{-3y+20}{2}\right) - 5y &= 11 && \text{multiply by 2} \\ 3(-3y+20) - 10y &= 22 && \text{distribute 3} \\ -9y + 60 - 10y &= 22 && \text{combine like terms} \\ -19y + 60 &= 22 && \text{subtract 60} \\ -19y &= -38 && \text{divide by } -19 \\ y &= 2 \end{aligned}$$

Now that we know that y is 2, we find x using the expression we used for the substitution.

$$x = \frac{-3y+20}{2} = \frac{-3\cdot 2+20}{2} = \frac{-6+20}{2} = \frac{14}{2} = 7$$

Thus, the solution of this system is $x = 7$ and $y = 2$, or, in short, $(7, 2)$. We check: the solution of a system is a simultaneous solution of both equations.

Checking $3x - 5y = 11$

$$\begin{aligned} \text{LHS} &= 3\cdot 7 - 5\cdot 2 = 21 - 10 = 11 \\ \text{RHS} &= 11 \checkmark \end{aligned}$$

Checking $2x + 3y = 20$

$$\begin{aligned} \text{LHS} &= 2\cdot 7 + 3\cdot 2 = 14 + 6 = 20 \\ \text{RHS} &= 20 \checkmark \end{aligned}$$

Therefore, our solution, $\boxed{(7, 2)}$ is correct.

Most real-world problems boil down to systems of equations. In this sense, solving systems of equations is one of the most important tasks in problem solving.

Example 3. There is an animal farm where chickens and cows live. All together, there are 53 heads and 174 legs. How many chickens and how many cows are there on the farm?

Solution: We will denote the number of chickens by x and the number of cows by y. The first equation will express the number of heads. x many chickens come with x many heads, and y many cows come with y many heads. The second equation will express the number of legs. x many chickens come with $2x$ many legs, and y many cows come with $4y$ many heads.

$$\begin{cases} x + y = 53 \\ 2x + 4y = 174 \end{cases}$$

Before we start solving the system, let us notice that we can simplify the second equation by dividing both sides by 2. We can often make our life easier with simplifications such as this one.

$$\begin{cases} x + y = 53 \\ x + 2y = 87 \end{cases}$$

We will solve for x in the first equation and substitute that expression into the second equation.

$$x = 53 - y \qquad \Longrightarrow \qquad (53 - y) + 2y = 87$$

We solve this equation for y.

$$\begin{aligned} 53 - y + 2y &= 87 \\ 53 + y &= 87 \qquad \text{subtract } 53 \\ y &= 34 \end{aligned}$$

Now that we know the value of y, we can easily find x.

$$x = 53 - 34 = 19 \qquad \Longrightarrow x = 19, \ \ y = 34$$

Thus we have $\boxed{\text{19 chickens and 34 cows}}$. We check: the number of heads is $19 + 34 = 53$, and the number of legs is $2 \cdot 19 + 4 \cdot 34 = 38 + 136 = 174$. So our solution is correct.

Example 4. We invested \$10 000 into two bank accounts. One account earns 14% per year, the other account earns 8% per year. How much did we invest into each account if after the first year, the combined interest from the two accounts is \$1238?

Solution: Let us denote the amount invested at 14% by x and the amount invested at 8% by y. The two equations will express the total amount invested, and the total interest earned. Then the interest earned form the first account is 14% of x, and that of the second account is 8% of y. Recall that 14% of x can be written as $0.14x$ and 8% of y as $0.08y$.

$$\begin{cases} x + y = 10\,000 & \text{the amounts invested add up to \$10 000} \\ 0.14x + 0.08y = 1238 & \text{the interests earned add up to \$1238} \end{cases}$$

We solve the system of equation by substitution, but let us first make the second equation simpler:

$$\begin{aligned} 0.14x + 0.08y &= 1238 \qquad \text{multiply by } 100 \\ 14x + 8y &= 123\,800 \quad \text{divide by } 2 \\ 7x + 4y &= 61\,900 \end{aligned}$$

We now have

$$\begin{cases} x + y = 10\,000 \\ 7x + 4y = 61\,900 \end{cases}$$

We will solve for y in the first equation and substitute the result into the second equation.

$$x + y = 10\,000 \quad \Longrightarrow \quad y = 10\,000 - x$$

Now the equation $7x + 4y = 61\,900$ becomes $7x + 4(10\,000 - x) = 61\,900$. We can solve this equation for x.

$$\begin{aligned} 7x + 4(10\,000 - x) &= 61\,900 \qquad \text{distribute } 4 \\ 7x + 40\,000 - 4x &= 61\,900 \qquad \text{combine like terms} \\ 3x + 40\,000 &= 61\,900 \qquad \text{subtract } 40\,000 \\ 3x &= 21\,900 \qquad \text{divide by } 3 \\ x &= 7300 \end{aligned}$$

We can now easily find y using $y = 10\,000 - x$.

$$y = 10\,000 - x = y = 10\,000 - 7300 = 2700$$

Our solution, $x = 7300$ and $y = 2700$ means that we invested $\boxed{\$7300 \text{ at } 14\% \text{ and } \$2700 \text{ at } 8\%}$. We check: the amounts add up to $\$7300 + \$2700 = \$10\,000$. The interest from the accounts are:

$$14\% \text{ of } 7300 \text{ is } 0.14\,(7300) = 1022 \text{ and } 8\% \text{ of } 2700 \text{ is } 0.08\,(2700) = 216$$

Since $1022 + 216 = 1238$, our solution is correct.

Example 5. We have a jar of coins, all pennies and dimes. All together, we have 372 coins, and the total value of all coins in the jar is $\$20.91$. How many pennies are there in the jar?

Solution: Let us denote the number of pennies by x and the number of dimes by y. The first equation will express the number of the coins. This equation is therefore $x + y = 372$. To express the value of all coins, x many pennies are worth $0.01x$ and y many dimes are worth $0.1y$. The total value of all coins is then

$$0.01x + 0.1y = 20.91$$

In order to clear the decimals, we may multiply both sides by 100. Then we have

$$x + 10y = 2091$$

Let us notice that this is the same equation that we would obtain if we expressed the value of all coins in cents and not in dollars. So our system is now

$$\begin{cases} x + y = 372 \\ x + 10y = 2091 \end{cases}$$

We solve for x in the first equation and substitute that into the second equation.

$$x + y = 372 \quad \Longrightarrow \quad x = 372 - y$$

Now the other equation, $x + 10y = 2091$ becomes

$$\begin{aligned} 372 - y + 10y &= 2091 && \text{combine like terms} \\ 9y + 372 &= 2091 && \text{subtract } 372 \\ 9y &= 1719 && \text{divide by } 9 \\ y &= 191 && \Longrightarrow \quad x = 372 - y = 372 - 191 = 181 \end{aligned}$$

The solution $x = 181$, $y = 191$ means that we have $\boxed{181 \text{ pennies and } 191 \text{ dimes}}$. We check: the number of all coins is $181 + 191 = 372$, and the value of the coins is $0.01 \cdot 181 + 0.1 \cdot 191 = 1.81 + 19.1 = 20.91$. Thus our solution is correct.

Practice Problems

1. Solve each of the following system of linear equations.

a) $\begin{cases} 3x + y = -4 \\ x - 3y = -8 \end{cases}$

b) $\begin{cases} 5(p-1) - 2(q-1) = 22 \\ p - q = 8 \end{cases}$

c) $\begin{cases} a + 3b = 10 \\ 3b - 5a = 22 \end{cases}$

d) $\begin{cases} \dfrac{1}{2}x + \dfrac{1}{4}y = 5 \\ \dfrac{1}{2}y - \dfrac{1}{3}x = -6 \end{cases}$

e) $\begin{cases} 2x - y = 1 \\ 2(y-3) = 6(x-1) \end{cases}$

f) $\begin{cases} 2a + 3b = -16 \\ (a+3)^2 = a^2 + 2b + 27 \end{cases}$

g) $\begin{cases} 2x + 3y = 3 \\ 5x - 2y = 4 \end{cases}$

h) $\begin{cases} 3x - 2y = -8 \\ -2x + 3y = 12 \end{cases}$

i) $\begin{cases} 2r - 0.5s = -1.7 \\ 1.5r + s = 0.65 \end{cases}$

2. Given the equations of two straight lines, find both coordinates of all intersection points.

a) $2x - 5y = -41$ and $x + y = 4$

b) $x + y = -5$ and $2x - y = -7$

c) $y = \dfrac{3}{4}x - 2$ and $2y = x$

d) $5x - y = -35$ and $y = -\dfrac{3}{4}x + \dfrac{1}{2}$

e) $y = -\dfrac{2}{3}x + 7$ and $x + 2y = 6$

3. There is an animal farm where chickens and cows live. All together, there are 52 heads and 134 legs. How many chickens and how many cows are there on the farm?

4. We invested \$9700 into two bank accounts. One account earns 7% per year, the other account earns 12% per year. How much did we invest into each account if after the first year, the combined interest from the two accounts is \$1004?

5. We have 54 coins, all dimes and quarters, in the total value of \$10.05. How many quarters and how many dimes are there?

6. We invested \$7800 into two bank accounts. One account earns 9% per year, the other account earns 10% per year. How much did we invest into each account if after the first year we have a total of \$8549 in the accounts?

Answers

Practice Problems

1. a) $x = -2, y = 2$ b) $p = 3, q = -5$ c) $a = -2, b = 4$ d) $x = 12, y = -4$ e) $x = -1, y = -3$
f) $a = 1, b = -6$ g) $x = \dfrac{18}{19}, y = \dfrac{7}{19}$ h) $x = 0, y = 4$ i) $r = -0.5, s = 1.4$

2. a) $(-3, 7)$ b) $(-4, y - 1)$ c) $(8, 4)$ d) $(-6, 5)$ e) $(24, -9)$ 3. 37 chickens, 15 cows

4. \$3200 at 7% and \$6500 at 12% 5. 23 dimes and 31 quarters 6. \$3100 at 9% and \$4700 at 10%

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