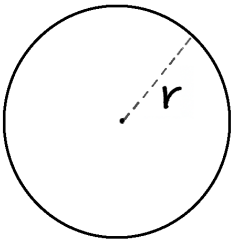
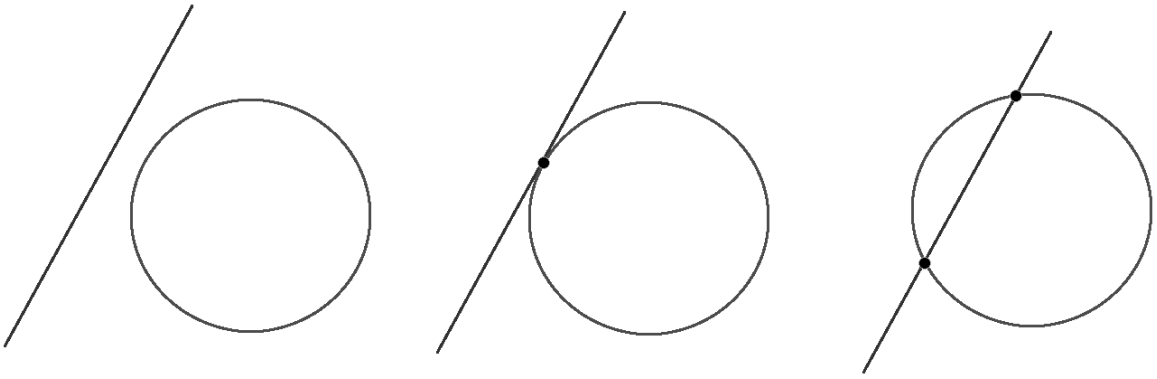
Recall the definition of a circle.

Definition: A circle is the set of all points in a plane that are equidistant to a fixed point. That equal distance is called the radius of the circle, that fixed point is called the center of the circle.



Part 1 - Tangent Lines

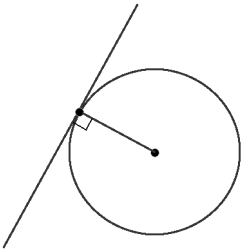
A circle and a line in the same plane could have no intersection points, one intersection point, or two intersection points. As it turns out, the case with the single intersection point is interesting and very important.



Definition: Suppose that a circle and a line in the same plane have exactly one intersection point. In this case, we say that the line is a **tangent line** of the circle. The point shared between the line and circle is called the **point of tangency**.

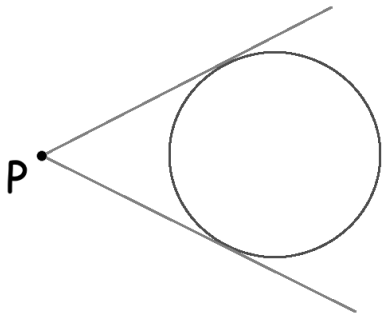
The following two theorems will be important. They often pop up in geometry proofs and problems.

Theorem: If a line is tangent to a circle, then the radius drawn to the point of tangency is perpendicular to the tangent line.



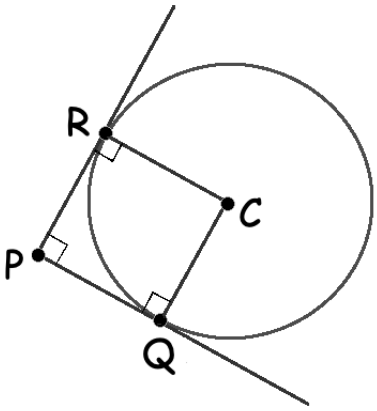
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If a point P is outside of the circle, (we also say that P is an external point), then there are two tangent lines from P to the circle.



Example 1. Given a circle, find all points P such that the tangent lines drawn to the circle from P are perpendicular to each other.

Solution: Let P be a point so that the tangent lines through P are perpendicular to each other. Let us label the center of the circle as C and the radius of the circle by r. Let Q and R be the points of tangency.

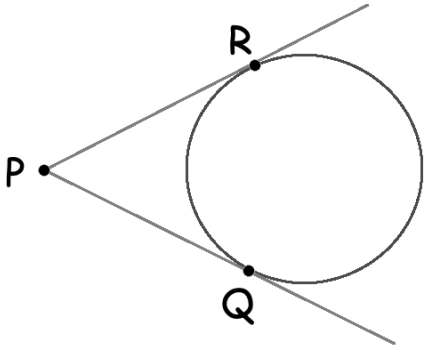


Consider now the quadrilateral $PQCR$. It is a rectangle because three of its angles are 90°. Then the fourth angle is also a right angle, and so we have a rectangle. Furthermore, it is a square because RC and CQ are equally long because they are radii in the same circle. So all four sides are of length r.

So P is always the vertex opposite to C in a square with sides r. Therefore the distance of P and C is the length of the diagonal of a square with sides r, and this distance is the same for every such point P.

So the points P with the required property are all on a circle, centered at C, whose radius is the diagonal of a square with sides r, where r is the radius of the circle.

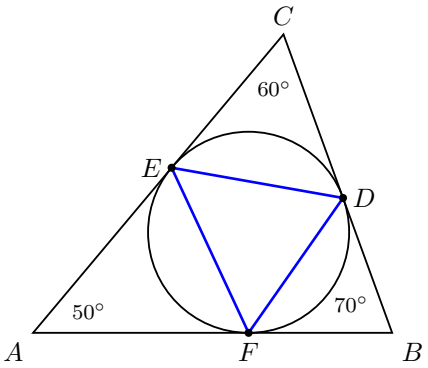
Theorem: (Tangent Segments Theorem) Consider P, a point external to a circle. Suppose we drew both tangent lines from P to the circle. If the points of tangency are Q and R, then $\overline{PQ} = \overline{PR}$.



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Example 2. Two angles of a triangle measure 50° and 70°. We draw a circle inside the triangle so that all three sides of the triangle are tangent lines to the circle. (This circle is called the inscribed circle of the triangle.) The three points of tangency form a new triangle. Find the angles of this new triangle.

Solution: The third angle of the triangle is $180° - 50° - 70° = 60°$. Let us label the vertices of the triangle by A, B, and C so that the angles at A, B, and C are 50°, 70°, and 60°. Let D, E, and F be the points of tangency on the sides BC, CA, and AB.



By the Tangent Segments Theorem, $\overline{AE} = \overline{AF}$, so the triangle AEF is isosceles. Its two base angles are equal, and so each of them measures $(180° - 50°)/2 = 65°$.

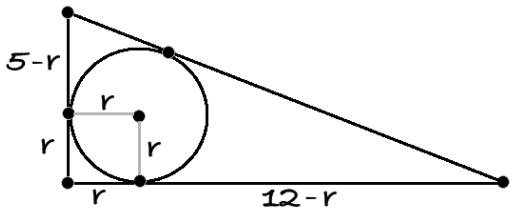
Similarly, $\overline{BD} = \overline{BF}$, so the base angles of the isosceles triangle BDF measure $(180° - 70°)/2 = 55°$. Also, $\overline{CD} = \overline{CE}$, so the base angles of the isosceles triangle CDE measure $(180° - 60°)/2 = 60°$.

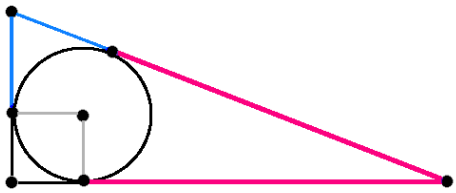
At each point of tangency, three angles together form a straight angle, 180°. At F, the angles $\angle AFE$, $\angle EFD$, and $\angle DFB$ form a straight angle, and so $\angle EFD = 180° - 65° - 55° = 60°$. In the same way, $\angle FDE = 180° - 55° - 60° = 65°$ at D, and $\angle DEF = 180° - 60° - 65° = 55°$ at E.

So the angles of the triangle formed by the points of tangency are 60°, 65°, and 55°. Indeed, these add up to 180°.

Example 3. For every triangle, there exists a unique circle inside the triangle such that all three sides are tangents to the circle. Such a circle is called the inscribed circle. Consider the right triangle with sides 5, 12, and 13 unit long. Find the radius of its inscribed circle.

Solution: The vertex opposite the hypotenuse, the center of the circle and the points of tangency on the shorter sides form a rectangle because three of its angles measure 90°. This rectangle is actually a square with sides r.





The point of tangency splits the shortest side into line segments r and $5 - r$ long. The point of tangency splits the second longest side into line segments r and $12 - r$ long.

If two tangent lines have a common external point, then the tangent line segments are equally long. This enables us to write a surprisingly simple equation expressing the length of the hypotenuse.

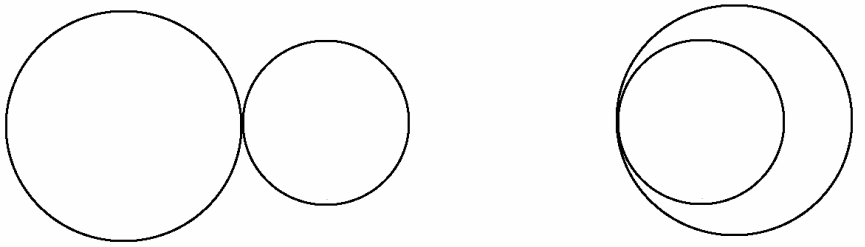
$$5 - r + 12 - r = 13$$

We solve the equation and get $r = 2$.

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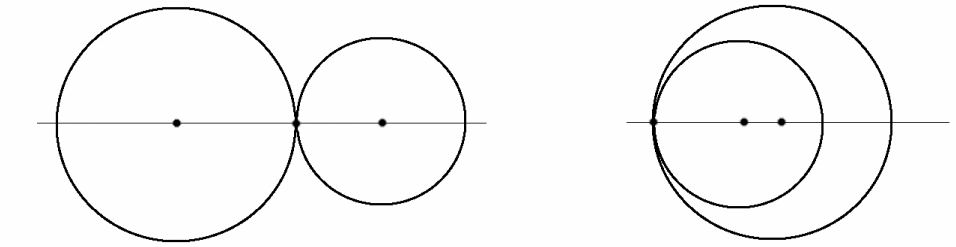
Part 2 - Circles Tangent to Each Other

Circles can also be tangent to each other. They can be internally or externally tangent.



> **Theorem:** If two circles are tangents to each other (externally or internally), then the line connecting the centers will pass through the point of tangency.

This fact is intuitive for externally tangent circles, but it might be less obvious for internally tangent circles.



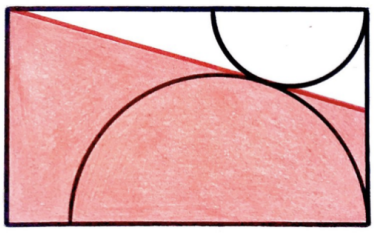


Practice Problems

1. Suppose that two circles are externally tangent to each other. Let P be the point of tangency. Let Q be any point on the line tangent to both circles and containing P. Prove that the tangent segments drawn from Q to the two circles are equally long.

The following puzzle is from the great Catriona Shearer.

2. What fraction of the rectangle is shaded?



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