

Chapter 6 : Factoring

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Factoring - Greatest Common Factor

Objective: Find the greatest common factor of a polynomial and factor it out of the expression.

The opposite of multiplying polynomials together is factoring polynomials. There are many benefits of a polynomial being factored. We use factored polynomials to help us solve equations, learn behaviors of graphs, work with fractions and more. Because so many concepts in algebra depend on us being able to factor polynomials it is very important to have very strong factoring skills.

In this lesson we will focus on factoring using the greatest common factor or GCF of a polynomial. When we multiplied polynomials, we multiplied monomials by polynomials by distributing, solving problems such as $4x^2(2x^2 - 3x + 8) = 8x^4 - 12x^3 + 32x$. In this lesson we will work the same problem backwards. We will start with $8x^2 - 12x^3 + 32x$ and try and work backwards to the $4x^2(2x - 3x + 8)$.

To do this we have to be able to first identify what is the GCF of a polynomial. We will first introduce this by looking at finding the GCF of several numbers. To find a GCF of several numbers we are looking for the largest number that can be divided by each of the numbers. This can often be done with quick mental math and it is shown in the following example

Example 262.

Find the GCF of 15, 24, and 27

$$\frac{15}{3} = 5, \quad \frac{24}{3} = 8, \quad \frac{27}{3} = 9 \quad \text{Each of the numbers can be divided by 3}$$

$$\text{GCF} = 3 \quad \text{Our Solution}$$

When there are variables in our problem we can first find the GCF of the num-

bers using mental math, then we take any variables that are in common with each term, using the lowest exponent. This is shown in the next example

Example 263.

$$\begin{array}{lcl} \text{GCF of } 24x^4y^2z, 18x^2y^4, \text{ and } 12x^3yz^5 & & \\ \frac{24}{6} = 4, \frac{18}{6} = 3, \frac{12}{6} = 2 & \text{Each number can be divided by 6} & \\ x^2y & x \text{ and } y \text{ are in all 3, using lowest exponents} & \\ \text{GCF} = 6x^2y & \text{Our Solution} & \end{array}$$

To factor out a GCF from a polynomial we first need to identify the GCF of all the terms, this is the part that goes in front of the parenthesis, then we divide each term by the GCF, the answer is what is left inside the parenthesis. This is shown in the following examples

Example 264.

$$\begin{array}{lcl} 4x^2 - 20x + 16 & \text{GCF is 4, divide each term by 4} & \\ \frac{4x^2}{4} = x^2, \frac{-20x}{4} = -5x, \frac{16}{4} = 4 & \text{This is what is left inside the parenthesis} & \\ 4(x^2 - 5x + 4) & \text{Our Solution} & \end{array}$$

With factoring we can always check our solutions by multiplying (distributing in this case) out the answer and the solution should be the original equation.

Example 265.

$$\begin{array}{lcl} 25x^4 - 15x^3 + 20x^2 & \text{GCF is } 5x^2, \text{ divide each term by this} & \\ \frac{25x^4}{5x^2} = 5x^2, \frac{-15x^3}{5x^2} = -3x, \frac{20x^2}{5x^2} = 4 & \text{This is what is left inside the parenthesis} & \\ 5x^2(5x^2 - 3x + 4) & \text{Our Solution} & \end{array}$$

Example 266.

$$3x^3y^2z + 5x^4y^3z^5 - 4xy^4 \quad \text{GCF is } xy^2, \text{ divide each term by this}$$

$$\frac{3x^3y^2z}{xy^2} = 3x^2z, \frac{5x^4y^3z^5}{xy^2} = 5x^3yz^5, \frac{-4xy^4}{xy^2} = -4y^2 \quad \text{This is what is left in parenthesis}$$

$$xy^2(3x^2z + 5x^3yz^5 - 4y^2) \quad \text{Our Solution}$$

World View Note: The first recorded algorithm for finding the greatest common factor comes from Greek mathematician Euclid around the year 300 BC!

Example 267.

$$\frac{21x^3}{7x} = 3x^2, \frac{14x^2}{7x} = 2x, \frac{7x}{7x} = 1 \quad \text{GCF is } 7x, \text{ divide each term by this}$$

$$7x(3x^2 + 2x + 1) \quad \text{This is what is left inside the parenthesis}$$

$$7x(3x^2 + 2x + 1) \quad \text{Our Solution}$$

It is important to note in the previous example, that when the GCF was $7x$ and $7x$ was one of the terms, dividing gave an answer of 1. Students often try to factor out the $7x$ and get zero which is incorrect, factoring will never make terms disappear. Anything divided by itself is 1, be sure to not forget to put the 1 into the solution.

Often the second line is not shown in the work of factoring the GCF. We can simply identify the GCF and put it in front of the parenthesis as shown in the following two examples.

Example 268.

$$12x^5y^2 - 6x^4y^4 + 8x^3y^5 \quad \text{GCF is } 2x^3y^2, \text{ put this in front of parenthesis and divide}$$

$$2x^3y^2(6x^2 - 3xy^2 + 4y^3) \quad \text{Our Solution}$$

Example 269.

$$18a^4b^3 - 27a^3b^3 + 9a^2b^3 \quad \text{GCF is } 9a^2b^3, \text{ divide each term by this}$$

$$9a^2b^3(2a^2 - 3a + 1) \quad \text{Our Solution}$$

Again, in the previous problem, when dividing $9a^2b^3$ by itself, the answer is 1, not zero. Be very careful that each term is accounted for in your final solution.

6.1 Practice - Greatest Common Factor

Factor the common factor out of each expression.

1) $9 + 8b^2$

2) $x - 5$

3) $45x^2 - 25$

4) $1 + 2n^2$

5) $56 - 35p$

6) $50x - 80y$

7) $7ab - 35a^2b$

8) $27x^2y^5 - 72x^3y^2$

9) $-3a^2b + 6a^3b^2$

10) $8x^3y^2 + 4x^3$

11) $-5x^2 - 5x^3 - 15x^4$

12) $-32n^9 + 32n^6 + 40n^5$

13) $20x^4 - 30x + 30$

14) $21p^6 + 30p^2 + 27$

15) $28m^4 + 40m^3 + 8$

16) $-10x^4 + 20x^2 + 12x$

17) $30b^9 + 5ab - 15a^2$

18) $27y^7 + 12y^2x + 9y^2$

19) $-48a^2b^2 - 56a^3b - 56a^5b$

20) $30m^6 + 15mn^2 - 25$

21) $20x^8y^2z^2 + 15x^5y^2z + 35x^3y^3z$

22) $3p + 12q - 15q^2r^2$

24) $30y^4z^3x^5 + 50y^4z^5 - 10y^4z^3x$

23) $50x^2y + 10y^2 + 70xz^2$

26) $28b + 14b^2 + 35b^3 + 7b^5$

25) $30qpr - 5qp + 5q$

28) $30a^8 + 6a^5 + 27a^3 + 21a^2$

27) $-18n^5 + 3n^3 - 21n + 3$

30) $-24x^6 - 4x^4 + 12x^3 + 4x^2$

29) $-40x^{11} - 20x^{12} + 50x^{13} - 50x^{14}$

32) $-10y^7 + 6y^{10} - 4y^{10}x - 8y^8x$

31) $-32mn^8 + 4m^6n + 12mn^4 + 16mn$

Factoring - Grouping

Objective: Factor polynomials with four terms using grouping.

The first thing we will always do when factoring is try to factor out a GCF. This GCF is often a monomial like in the problem $5xy + 10xz$ the GCF is the monomial $5x$, so we would have $5x(y + 2z)$. However, a GCF does not have to be a monomial, it could be a binomial. To see this, consider the following two example.

Example 270.

$$\begin{array}{ll} 3ax - 7bx & \text{Both have } x \text{ in common, factor it out} \\ x(3a - 7b) & \text{Our Solution} \end{array}$$

Now the same problem, but instead of x we have $(2a + 5b)$.

Example 271.

$$\begin{array}{ll} 3a(2a + 5b) - 7b(2a + 5b) & \text{Both have } (2a + 5b) \text{ in common, factor it out} \\ (2a + 5b)(3a - 7b) & \text{Our Solution} \end{array}$$

In the same way we factored out a GCF of x we can factor out a GCF which is a binomial, $(2a + 5b)$. This process can be extended to factor problems where there is no GCF to factor out, or after the GCF is factored out, there is more factoring that can be done. Here we will have to use another strategy to factor. We will use a process known as grouping. Grouping is how we will factor if there are four terms in the problem. Remember, factoring is like multiplying in reverse, so first we will look at a multiplication problem and then try to reverse the process.

Example 272.

$$\begin{array}{ll} (2a + 3)(5b + 2) & \text{Distribute } (2a + 3) \text{ into second parenthesis} \\ 5b(2a + 3) + 2(2a + 3) & \text{Distribute each monomial} \\ 10ab + 15b + 4a + 6 & \text{Our Solution} \end{array}$$

The solution has four terms in it. We arrived at the solution by looking at the two parts, $5b(2a + 3)$ and $2(2a + 3)$. When we are factoring by grouping we will always divide the problem into two parts, the first two terms and the last two terms. Then we can factor the GCF out of both the left and right sides. When we do this our hope is what is left in the parenthesis will match on both the left and right. If they match we can pull this matching GCF out front, putting the rest in parenthesis and we will be factored. The next example is the same problem worked backwards, factoring instead of multiplying.

Example 273.

$$\begin{array}{ll}
10ab + 15b + 4a + 6 & \text{Split problem into two groups} \\
\boxed{10ab + 15b} \mid \boxed{+ 4a + 6} & \text{GCF on left is } 5b, \text{ on the right is } 2 \\
\boxed{5b(2a + 3)} \mid \boxed{+ 2(2a + 3)} & (2a + 3) \text{ is the same! Factor out this GCF} \\
(2a + 3)(5b + 2) & \text{Our Solution}
\end{array}$$

The key for grouping to work is after the GCF is factored out of the left and right, the two binomials must match exactly. If there is any difference between the two we either have to do some adjusting or it can't be factored using the grouping method. Consider the following example.

Example 274.

$$\begin{array}{ll}
6x^2 + 9xy - 14x - 21y & \text{Split problem into two groups} \\
\boxed{6x^2 + 9xy} \mid \boxed{- 14x - 21y} & \text{GCF on left is } 3x, \text{ on right is } 7 \\
\boxed{3x(2x + 3y)} \mid \boxed{+ 7(- 2x - 3y)} & \text{The signs in the parenthesis don't match!}
\end{array}$$

when the signs don't match on both terms we can easily make them match by factoring the opposite of the GCF on the right side. Instead of 7 we will use -7 . This will change the signs inside the second parenthesis.

$$\begin{array}{ll}
\boxed{3x(2x + 3y)} \mid \boxed{- 7(2x + 3y)} & (2x + 3y) \text{ is the same! Factor out this GCF} \\
(2x + 3y)(3x - 7) & \text{Our Solution}
\end{array}$$

Often we can recognize early that we need to use the opposite of the GCF when factoring. If the first term of the first binomial is positive in the problem, we will also want the first term of the second binomial to be positive. If it is negative then we will use the opposite of the GCF to be sure they match.

Example 275.

$$\begin{array}{ll}
5xy - 8x - 10y + 16 & \text{Split the problem into two groups} \\
\boxed{5xy - 8x} \mid \boxed{- 10y + 16} & \text{GCF on left is } x, \text{ on right we need a negative,} \\
& \text{so we use } -2 \\
\boxed{x(5y - 8)} \mid \boxed{- 2(5y - 8)} & (5y - 8) \text{ is the same! Factor out this GCF} \\
(5y - 8)(x - 2) & \text{Our Solution}
\end{array}$$

Sometimes when factoring the GCF out of the left or right side there is no GCF to factor out. In this case we will use either the GCF of 1 or -1 . Often this is all we need to be sure the two binomials match.

Example 276.

$12ab - 14a - 6b + 7$	Split the problem into two groups
$\boxed{12ab - 14a} \quad \boxed{-6b + 7}$	GCF on left is $2a$, on right, no GCF, use -1
$\boxed{2a(6b - 7)} \quad \boxed{-1(6b - 7)}$	$(6b - 7)$ is the same! Factor out this GCF
$(6b - 7)(2a - 1)$	Our Solution

Example 277.

$6x^3 - 15x^2 + 2x - 5$	Split problem into two groups
$\boxed{6x^3 - 15x^2} \quad \boxed{+ 2x - 5}$	GCF on left is $3x^2$, on right, no GCF, use 1
$\boxed{3x^2(2x - 5)} \quad \boxed{+ 1(2x - 5)}$	$(2x - 5)$ is the same! Factor out this GCF
$(2x - 5)(3x^2 + 1)$	Our Solution

Another problem that may come up with grouping is after factoring out the GCF on the left and right, the binomials don't match, more than just the signs are different. In this case we may have to adjust the problem slightly. One way to do this is to change the order of the terms and try again. To do this we will move the second term to the end of the problem and see if that helps us use grouping.

Example 278.

$4a^2 - 21b^3 + 6ab - 14ab^2$	Split the problem into two groups
$\boxed{4a^2 - 21b^3} \quad \boxed{+ 6ab - 14ab^2}$	GCF on left is 1, on right is $2ab$
$\boxed{1(4a^2 - 21b^3)} \quad \boxed{+ 2ab(3 - 7b)}$	Binomials don't match! Move second term to end
$4a^2 + 6ab - 14ab^2 - 21b^3$	Start over, split the problem into two groups
$\boxed{4a^2 + 6ab} \quad \boxed{- 14ab^2 - 21b^3}$	GCF on left is $2a$, on right is $-7b^2$
$\boxed{2a(2a + 3b)} \quad \boxed{- 7b^2(2a + 3b)}$	$(2a + 3b)$ is the same! Factor out this GCF
$(2a + 3b)(2a - 7b^2)$	Our Solution

When rearranging terms the problem can still be out of order. Sometimes after factoring out the GCF the terms are backwards. There are two ways that this can happen, one with addition, one with subtraction. If it happens with addition, for

example the binomials are $(a + b)$ and $(b + a)$, we don't have to do any extra work. This is because addition is the same in either order ($5 + 3 = 3 + 5 = 8$).

Example 279.

$7 + y - 3xy - 21x$	Split the problem into two groups		
<table border="1" style="display: inline-table; border-collapse: collapse;"> <tr> <td style="padding: 2px 10px;">$7 + y$</td> <td style="padding: 2px 10px;">$- 3xy - 21x$</td> </tr> </table>	$7 + y$	$- 3xy - 21x$	GCF on left is 1, on the right is $- 3x$
$7 + y$	$- 3xy - 21x$		
<table border="1" style="display: inline-table; border-collapse: collapse;"> <tr> <td style="padding: 2px 10px;">$1(7 + y)$</td> <td style="padding: 2px 10px;">$- 3x(y + 7)$</td> </tr> </table>	$1(7 + y)$	$- 3x(y + 7)$	$y + 7$ and $7 + y$ are the same, use either one
$1(7 + y)$	$- 3x(y + 7)$		
$(y + 7)(1 - 3x)$	Our Solution		

However, if the binomial has subtraction, then we need to be a bit more careful. For example, if the binomials are $(a - b)$ and $(b - a)$, we will factor out the opposite of the GCF on one part, usually the second. Notice what happens when we factor out $- 1$.

Example 280.

$(b - a)$	Factor out $- 1$
$- 1(- b + a)$	Addition can be in either order, switch order
$- 1(a - b)$	The order of the subtraction has been switched!

Generally we won't show all the above steps, we will simply factor out the opposite of the GCF and switch the order of the subtraction to make it match the other binomial.

Example 281.

$8xy - 12y + 15 - 10x$	Split the problem into two groups		
<table border="1" style="display: inline-table; border-collapse: collapse;"> <tr> <td style="padding: 2px 10px;">$8xy - 12y$</td> <td style="padding: 2px 10px;">$15 - 10x$</td> </tr> </table>	$8xy - 12y$	$15 - 10x$	GCF on left is $4y$, on right, 5
$8xy - 12y$	$15 - 10x$		
<table border="1" style="display: inline-table; border-collapse: collapse;"> <tr> <td style="padding: 2px 10px;">$4y(2x - 3)$</td> <td style="padding: 2px 10px;">$+ 5(3 - 2x)$</td> </tr> </table>	$4y(2x - 3)$	$+ 5(3 - 2x)$	Need to switch subtraction order, use $- 5$ in middle
$4y(2x - 3)$	$+ 5(3 - 2x)$		
<table border="1" style="display: inline-table; border-collapse: collapse;"> <tr> <td style="padding: 2px 10px;">$4y(2y - 3)$</td> <td style="padding: 2px 10px;">$- 5(2x - 3)$</td> </tr> </table>	$4y(2y - 3)$	$- 5(2x - 3)$	Now $2x - 3$ match on both! Factor out this GCF
$4y(2y - 3)$	$- 5(2x - 3)$		
$(2x - 3)(4y - 5)$	Our Solution		

World View Note: Sofia Kovalevskaya of Russia was the first woman on the editorial staff of a mathematical journal in the late 19th century. She also did research on how the rings of Saturn rotated.

6.2 Practice - Grouping

Factor each completely.

1) $40r^3 - 8r^2 - 25r + 5$

2) $35x^3 - 10x^2 - 56x + 16$

3) $3n^3 - 2n^2 - 9n + 6$

4) $14v^3 + 10v^2 - 7v - 5$

5) $15b^3 + 21b^2 - 35b - 49$

6) $6x^3 - 48x^2 + 5x - 40$

7) $3x^3 + 15x^2 + 2x + 10$

8) $28p^3 + 21p^2 + 20p + 15$

9) $35x^3 - 28x^2 - 20x + 16$

10) $7n^3 + 21n^2 - 5n - 15$

11) $7xy - 49x + 5y - 35$

12) $42r^3 - 49r^2 + 18r - 21$

13) $32xy + 40x^2 + 12y + 15x$

14) $15ab - 6a + 5b^3 - 2b^2$

15) $16xy - 56x + 2y - 7$

16) $3mn - 8m + 15n - 40$

17) $2xy - 8x^2 + 7y^3 - 28y^2x$

18) $5mn + 2m - 25n - 10$

19) $40xy + 35x - 8y^2 - 7y$

20) $8xy + 56x - y - 7$

21) $32uv - 20u + 24v - 15$

22) $4uv + 14u^2 + 12v + 42u$

23) $10xy + 30 + 25x + 12y$

24) $24xy + 25y^2 - 20x - 30y^3$

25) $3uv + 14u - 6u^2 - 7v$

26) $56ab + 14 - 49a - 16b$

27) $16xy - 3x - 6x^2 + 8y$

Factoring - Trinomials where $a = 1$

Objective: Factor trinomials where the coefficient of x^2 is one.

Factoring with three terms, or trinomials, is the most important type of factoring to be able to master. As factoring is multiplication backwards we will start with a multiplication problem and look at how we can reverse the process.

Example 282.

$$\begin{array}{ll}
 (x+6)(x-4) & \text{Distribute } (x+6) \text{ through second parenthesis} \\
 x(x+6) - 4(x+6) & \text{Distribute each monomial through parenthesis} \\
 x^2 + 6x - 4x - 24 & \text{Combine like terms} \\
 x^2 + 2x - 24 & \text{Our Solution}
 \end{array}$$

You may notice that if you reverse the last three steps the process looks like grouping. This is because it is grouping! The GCF of the left two terms is x and the GCF of the second two terms is -4 . The way we will factor trinomials is to make them into a polynomial with four terms and then factor by grouping. This is shown in the following example, the same problem worked backwards

Example 283.

$$\begin{array}{ll}
 x^2 + 2x - 24 & \text{Split middle term into } +6x - 4x \\
 x^2 + 6x - 4x - 24 & \text{Grouping: GCF on left is } x, \text{ on right is } -4 \\
 x(x+6) - 4(x+6) & (x+6) \text{ is the same, factor out this GCF} \\
 (x+6)(x-4) & \text{Our Solution}
 \end{array}$$

The trick to make these problems work is how we split the middle term. Why did we pick $+6x - 4x$ and not $+5x - 3x$? The reason is because $6x - 4x$ is the only combination that works! So how do we know what is the one combination that works? To find the correct way to split the middle term we will use what is called the ac method. In the next lesson we will discuss why it is called the ac method. The way the ac method works is we find a pair of numbers that multiply to a certain number and add to another number. Here we will try to multiply to get the last term and add to get the coefficient of the middle term. In the previous

example that would mean we wanted to multiply to -24 and add to $+2$. The only numbers that can do this are 6 and -4 ($6 \cdot -4 = -24$ and $6 + (-4) = 2$). This process is shown in the next few examples

Example 284.

$$\begin{array}{ll} x^2 + 9x + 18 & \text{Want to multiply to 18, add to 9} \\ x^2 + 6x + 3x + 18 & 6 \text{ and } 3, \text{ split the middle term} \\ x(x + 6) + 3(x + 6) & \text{Factor by grouping} \\ (x + 6)(x + 3) & \text{Our Solution} \end{array}$$

Example 285.

$$\begin{array}{ll} x^2 - 4x + 3 & \text{Want to multiply to 3, add to } -4 \\ x^2 - 3x - x + 3 & -3 \text{ and } -1, \text{ split the middle term} \\ x(x - 3) - 1(x - 3) & \text{Factor by grouping} \\ (x - 3)(x - 1) & \text{Our Solution} \end{array}$$

Example 286.

$$\begin{array}{ll} x^2 - 8x - 20 & \text{Want to multiply to } -20, \text{ add to } -8 \\ x^2 - 10x + 2x - 20 & -10 \text{ and } 2, \text{ split the middle term} \\ x(x - 10) + 2(x - 10) & \text{Factor by grouping} \\ (x - 10)(x + 2) & \text{Our Solution} \end{array}$$

Often when factoring we have two variables. These problems solve just like problems with one variable, using the coefficients to decide how to split the middle term

Example 287.

$$\begin{array}{ll} a^2 - 9ab + 14b^2 & \text{Want to multiply to 14, add to } -9 \\ a^2 - 7ab - 2ab + 14b^2 & -7 \text{ and } -2, \text{ split the middle term} \\ a(a - 7b) - 2b(a - 7b) & \text{Factor by grouping} \\ (a - 7b)(a - 2b) & \text{Our Solution} \end{array}$$

As the past few examples illustrate, it is very important to be aware of negatives as we find the pair of numbers we will use to split the middle term. Consider the following example, done incorrectly, ignoring negative signs

Warning 288.

$$\begin{array}{ll}
 x^2 + 5x - 6 & \text{Want to multiply to 6, add 5} \\
 x^2 + 2x + 3x - 6 & \text{2 and 3, split the middle term} \\
 x(x + 2) + 3(x - 2) & \text{Factor by grouping} \\
 ??? & \text{Binomials do not match!}
 \end{array}$$

Because we did not use the negative sign with the six to find our pair of numbers, the binomials did not match and grouping was not able to work at the end. Now the problem will be done correctly.

Example 289.

$$\begin{array}{ll}
 x^2 + 5x - 6 & \text{Want to multiply to } -6, \text{ add to } 5 \\
 x^2 + 6x - x - 6 & \text{6 and } -1, \text{ split the middle term} \\
 x(x + 6) - 1(x + 6) & \text{Factor by grouping} \\
 (x + 6)(x - 1) & \text{Our Solution}
 \end{array}$$

You may have noticed a shortcut for factoring these problems. Once we identify the two numbers that are used to split the middle term, these are the two numbers in our factors! In the previous example, the numbers used to split the middle term were 6 and -1 , our factors turned out to be $(x + 6)(x - 1)$. This pattern does not always work, so be careful getting in the habit of using it. We can use it however, when we have no number (technically we have a 1) in front of x^2 . In all the problems we have factored in this lesson there is no number in front of x^2 . If this is the case then we can use this shortcut. This is shown in the next few examples.

Example 290.

$$\begin{array}{ll}
 x^2 - 7x - 18 & \text{Want to multiply to } -18, \text{ add to } -7 \\
 & -9 \text{ and } 2, \text{ write the factors} \\
 (x - 9)(x + 2) & \text{Our Solution}
 \end{array}$$

Example 291.

$$\begin{array}{ll}
m^2 - mn - 30n^2 & \text{Want to multiply to } -30, \text{ add to } -1 \\
& 5 \text{ and } -6, \text{ write the factors, don't forget second variable} \\
(m + 5n)(m - 6n) & \text{Our Solution}
\end{array}$$

It is possible to have a problem that does not factor. If there is no combination of numbers that multiplies and adds to the correct numbers, then we say we cannot factor the polynomial, or we say the polynomial is prime. This is shown in the following example.

Example 292.

$$\begin{array}{ll}
x^2 + 2x + 6 & \text{Want to multiply to 6, add to 2} \\
1 \cdot 6 \text{ and } 2 \cdot 3 & \text{Only possibilities to multiply to six, none add to 2} \\
\text{Prime, can't factor} & \text{Our Solution}
\end{array}$$

When factoring it is important not to forget about the GCF. If all the terms in a problem have a common factor we will want to first factor out the GCF before we factor using any other method.

Example 293.

$$\begin{array}{ll}
3x^2 - 24x + 45 & \text{GCF of all terms is 3, factor this out} \\
3(x^2 - 8x + 15) & \text{Want to multiply to 15, add to } -8 \\
& -5 \text{ and } -3, \text{ write the factors} \\
3(x - 5)(x - 3) & \text{Our Solution}
\end{array}$$

Again it is important to comment on the shortcut of jumping right to the factors, this only works if there is no coefficient on x^2 . In the next lesson we will look at how this process changes slightly when we have a number in front of x^2 . Be careful not to use this shortcut on all factoring problems!

World View Note: The first person to use letters for unknown values was Francois Vieta in 1591 in France. He used vowels to represent variables we are solving for, just as codes used letters to represent an unknown message.

6.3 Practice - Trinomials where a = 1

Factor each completely.

1) $p^2 + 17p + 72$

2) $x^2 + x - 72$

3) $n^2 - 9n + 8$

4) $x^2 + x - 30$

5) $x^2 - 9x - 10$

6) $x^2 + 13x + 40$

7) $b^2 + 12b + 32$

8) $b^2 - 17b + 70$

9) $x^2 + 3x - 70$

10) $x^2 + 3x - 18$

11) $n^2 - 8n + 15$

12) $a^2 - 6a - 27$

13) $p^2 + 15p + 54$

14) $p^2 + 7p - 30$

15) $n^2 - 15n + 56$

16) $m^2 - 15mn + 50n^2$

17) $u^2 - 8uv + 15v^2$

18) $m^2 - 3mn - 40n^2$

19) $m^2 + 2mn - 8n^2$

20) $x^2 + 10xy + 16y^2$

21) $x^2 - 11xy + 18y^2$

22) $u^2 - 9uv + 14v^2$

23) $x^2 + xy - 12y^2$

24) $x^2 + 14xy + 45y^2$

25) $x^2 + 4xy - 12y^2$

26) $4x^2 + 52x + 168$

27) $5a^2 + 60a + 100$

28) $5n^2 - 45n + 40$

29) $6a^2 + 24a - 192$

30) $5v^2 + 20v - 25$

31) $6x^2 + 18xy + 12y^2$

32) $5m^2 + 30mn - 90n^2$

33) $6x^2 + 96xy + 378y^2$

34) $6m^2 - 36mn - 162n^2$

Factoring - Trinomials where $a \neq 1$

Objective: Factor trinomials using the ac method when the coefficient of x^2 is not one.

When factoring trinomials we used the ac method to split the middle term and then factor by grouping. The ac method gets its name from the general trinomial equation, $ax^2 + bx + c$, where a , b , and c are the numbers in front of x^2 , x and the constant at the end respectively.

World View Note: It was French philosopher Rene Descartes who first used letters from the beginning of the alphabet to represent values we know (a , b , c) and letters from the end to represent letters we don't know and are solving for (x , y , z).

The ac method is named ac because we multiply $a \cdot c$ to find out what we want to multiply to. In the previous lesson we always multiplied to just c because there was no number in front of x^2 . This meant the number was 1 and we were multiplying to $1c$ or just c . Now we will have a number in front of x^2 so we will be looking for numbers that multiply to ac and add to b . Other than this, the process will be the same.

Example 294.

$3x^2 + 11x + 6$	Multiply to ac or $(3)(6) = 18$, add to 11
$3x^2 + 9x + 2x + 6$	The numbers are 9 and 2, split the middle term
$3x(x + 3) + 2(x + 3)$	Factor by grouping
$(x + 3)(3x + 2)$	Our Solution

When $a = 1$, or no coefficient in front of x^2 , we were able to use a shortcut, using the numbers that split the middle term in the factors. The previous example illustrates an important point, the shortcut does not work when $a \neq 1$. We must go through all the steps of grouping in order to factor the problem.

Example 295.

$8x^2 - 2x - 15$	Multiply to ac or $(8)(-15) = -120$, add to -2
$8x^2 - 12x + 10x - 15$	The numbers are -12 and 10 , split the middle term
$4x(2x - 3) + 5(2x - 3)$	Factor by grouping
$(2x - 3)(4x + 5)$	Our Solution

Example 296.

$10x^2 - 27x + 5$	Multiply to ac or $(10)(5) = 50$, add to -27
$10x^2 - 25x - 2x + 5$	The numbers are -25 and -2 , split the middle term
$5x(2x - 5) - 1(2x - 5)$	Factor by grouping
$(2x - 5)(5x - 1)$	Our Solution

The same process works with two variables in the problem

Example 297.

$4x^2 - xy - 5y^2$	Multiply to ac or $(4)(-5) = -20$, add to -1
$4x^2 + 4xy - 5xy - 5y^2$	The numbers are 4 and -5 , split the middle term
$4x(x + y) - 5y(x + y)$	Factor by grouping
$(x + y)(4x - 5y)$	Our Solution

As always, when factoring we will first look for a GCF before using any other method, including the ac method. Factoring out the GCF first also has the added bonus of making the numbers smaller so the ac method becomes easier.

Example 298.

$18x^3 + 33x^2 - 30x$	GCF = $3x$, factor this out first
$3x[6x^2 + 11x - 10]$	Multiply to ac or $(6)(-10) = -60$, add to 11
$3x[6x^2 + 15x - 4x - 10]$	The numbers are 15 and -4 , split the middle term
$3x[3x(2x + 5) - 2(2x + 5)]$	Factor by grouping
$3x(2x + 5)(3x - 2)$	Our Solution

As was the case with trinomials when $a = 1$, not all trinomials can be factored. If there is no combinations that multiply and add correctly then we can say the trinomial is prime and cannot be factored.

Example 299.

$3x^2 + 2x - 7$	Multiply to ac or $(3)(-7) = -21$, add to 2
$-3(7)$ and $-7(3)$	Only two ways to multiply to -21 , it doesn't add to 2
Prime, cannot be factored	Our Solution

6.4 Practice - Trinomials where $a \neq 1$

Factor each completely.

1) $7x^2 - 48x + 36$

2) $7n^2 - 44n + 12$

3) $7b^2 + 15b + 2$

4) $7v^2 - 24v - 16$

5) $5a^2 - 13a - 28$

6) $5n^2 - 4n - 20$

7) $2x^2 - 5x + 2$

8) $3r^2 - 4r - 4$

9) $2x^2 + 19x + 35$

10) $7x^2 + 29x - 30$

11) $2b^2 - b - 3$

12) $5k^2 - 26k + 24$

13) $5k^2 + 13k + 6$

14) $3r^2 + 16r + 21$

15) $3x^2 - 17x + 20$

16) $3u^2 + 13uv - 10v^2$

17) $3x^2 + 17xy + 10y^2$

18) $7x^2 - 2xy - 5y^2$

19) $5x^2 + 28xy - 49y^2$

20) $5u^2 + 31uv - 28v^2$

21) $6x^2 - 39x - 21$

22) $10a^2 - 54a - 36$

23) $21k^2 - 87k - 90$

24) $21n^2 + 45n - 54$

25) $14x^2 - 60x + 16$

26) $4r^2 + r - 3$

27) $6x^2 + 29x + 20$

28) $6p^2 + 11p - 7$

29) $4k^2 - 17k + 4$

30) $4r^2 + 3r - 7$

31) $4x^2 + 9xy + 2y^2$

32) $4m^2 + 6mn + 6n^2$

33) $4m^2 - 9mn - 9n^2$

34) $4x^2 - 6xy + 30y^2$

35) $4x^2 + 13xy + 3y^2$

36) $18u^2 - 3uv - 36v^2$

37) $12x^2 + 62xy + 70y^2$

38) $16x^2 + 60xy + 36y^2$

39) $24x^2 - 52xy + 8y^2$

40) $12x^2 + 50xy + 28y^2$

Factoring - Factoring Special Products

Objective: Identify and factor special products including a difference of squares, perfect squares, and sum and difference of cubes.

When factoring there are a few special products that, if we can recognize them, can help us factor polynomials. The first is one we have seen before. When multiplying special products we found that a sum and a difference could multiply to a difference of squares. Here we will use this special product to help us factor

$$\text{Difference of Squares: } a^2 - b^2 = (a + b)(a - b)$$

If we are subtracting two perfect squares then it will always factor to the sum and difference of the square roots.

Example 300.

$$\begin{array}{ll} x^2 - 16 & \text{Subtracting two perfect squares, the square roots are } x \text{ and } 4 \\ (x + 4)(x - 4) & \text{Our Solution} \end{array}$$

Example 301.

$$\begin{array}{ll} 9a^2 - 25b^2 & \text{Subtracting two perfect squares, the square roots are } 3a \text{ and } 5b \\ (3a + 5b)(3a - 5b) & \text{Our Solution} \end{array}$$

It is important to note, that a sum of squares will never factor. It is always prime. This can be seen if we try to use the ac method to factor $x^2 + 36$.

Example 302.

$$\begin{array}{ll} x^2 + 36 & \text{No } bx \text{ term, we use } 0x. \\ x^2 + 0x + 36 & \text{Multiply to 36, add to 0} \\ 1 \cdot 36, 2 \cdot 18, 3 \cdot 12, 4 \cdot 9, 6 \cdot 6 & \text{No combinations that multiply to 36 add to 0} \\ \text{Prime, cannot factor} & \text{Our Solution} \end{array}$$

It turns out that a sum of squares is always prime.

Sum of Squares: $a^2 + b^2 = \text{Prime}$

A great example where we see a sum of squares comes from factoring a difference of 4th powers. Because the square root of a fourth power is a square ($\sqrt{a^4} = a^2$), we can factor a difference of fourth powers just like we factor a difference of squares, to a sum and difference of the square roots. This will give us two factors, one which will be a prime sum of squares, and a second which will be a difference of squares which we can factor again. This is shown in the following examples.

Example 303.

$a^4 - b^4$	Difference of squares with roots a^2 and b^2
$(a^2 + b^2)(a^2 - b^2)$	The first factor is prime, the second is a difference of squares!
$(a^2 + b^2)(a + b)(a - b)$	Our Solution

Example 304.

$x^4 - 16$	Difference of squares with roots x^2 and 4
$(x^2 + 4)(x^2 - 4)$	The first factor is prime, the second is a difference of squares!
$(x^2 + 4)(x + 2)(x - 2)$	Our Solution

Another factoring shortcut is the perfect square. We had a shortcut for multiplying a perfect square which can be reversed to help us factor a perfect square

Perfect Square: $a^2 + 2ab + b^2 = (a + b)^2$

A perfect square can be difficult to recognize at first glance, but if we use the ac method and get two of the same numbers we know we have a perfect square. Then we can just factor using the square roots of the first and last terms and the sign from the middle. This is shown in the following examples.

Example 305.

$x^2 - 6x + 9$	Multiply to 9, add to -6
	The numbers are -3 and -3 , the same! Perfect square
$(x - 3)^2$	Use square roots from first and last terms and sign from the middle

Example 306.

$$4x^2 + 20xy + 25y^2 \quad \begin{array}{l} \text{Multiply to 100, add to 20} \\ \text{The numbers are 10 and 10, the same! Perfect square} \end{array}$$

$$(2x + 5y)^2 \quad \text{Uses square roots from first and last terms and sign from the middle}$$

World View Note: The first known record of work with polynomials comes from the Chinese around 200 BC. Problems would be written as “three sheafs of a good crop, two sheafs of a mediocre crop, and one sheaf of a bad crop sold for 29 dou. This would be the polynomial (trinomial) $3x + 2y + z = 29$.

Another factoring shortcut has cubes. With cubes we can either do a sum or a difference of cubes. Both sum and difference of cubes have very similar factoring formulas

$$\text{Sum of Cubes: } a^3 + b^3 = (a + b)(a^2 - ab + b^2)$$

$$\text{Difference of Cubes: } a^3 - b^3 = (a - b)(a^2 + ab + b^2)$$

Comparing the formulas you may notice that the only difference is the signs in between the terms. One way to keep these two formulas straight is to think of SOAP. S stands for Same sign as the problem. If we have a sum of cubes, we add first, a difference of cubes we subtract first. O stands for Opposite sign. If we have a sum, then subtraction is the second sign, a difference would have addition for the second sign. Finally, AP stands for Always Positive. Both formulas end with addition. The following examples show factoring with cubes.

Example 307.

$$m^3 - 27 \quad \begin{array}{l} \text{We have cube roots } m \text{ and } 3 \end{array}$$

$$(m - 3)(m^2 + 3m + 9) \quad \text{Use formula, use SOAP to fill in signs}$$

$$(m - 3)(m^2 + 3m + 9) \quad \text{Our Solution}$$

Example 308.

$$125p^3 + 8r^3 \quad \begin{array}{l} \text{We have cube roots } 5p \text{ and } 2r \end{array}$$

$$(5p + 2r)(25p^2 - 10r + 4r^2) \quad \text{Use formula, use SOAP to fill in signs}$$

$$(5p + 2r)(25p^2 - 10r + 4r^2) \quad \text{Our Solution}$$

The previous example illustrates an important point. When we fill in the trinomial's first and last terms we square the cube roots $5p$ and $2r$. Often students forget to square the number in addition to the variable. Notice that when done correctly, both get cubed.

Often after factoring a sum or difference of cubes, students want to factor the second factor, the trinomial further. As a general rule, this factor will always be prime (unless there is a GCF which should have been factored out before using cubes rule).

The following table summarizes all of the shortcuts that we can use to factor special products

Factoring Special Products

Difference of Squares	$a^2 - b^2 = (a + b)(a - b)$
Sum of Squares	$a^2 + b^2 = \text{Prime}$
Perfect Square	$a^2 + 2ab + b^2 = (a + b)^2$
Sum of Cubes	$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$
Difference of Cubes	$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$

As always, when factoring special products it is important to check for a GCF first. Only after checking for a GCF should we be using the special products. This is shown in the following examples

Example 309.

$$\begin{array}{ll}
 72x^2 - 2 & \text{GCF is 2} \\
 2(36x^2 - 1) & \text{Difference of Squares, square roots are } 6x \text{ and } 1 \\
 2(6x + 1)(6x - 1) & \text{Our Solution}
 \end{array}$$

Example 310.

$$\begin{array}{ll}
 48x^2y - 24xy + 3y & \text{GCF is } 3y \\
 3y(16x^2 - 8x + 1) & \text{Multiply to 16 add to 8} \\
 & \text{The numbers are 4 and 4, the same! Perfect Square} \\
 3y(4x - 1)^2 & \text{Our Solution}
 \end{array}$$

Example 311.

$$\begin{array}{ll}
 128a^4b^2 + 54ab^5 & \text{GCF is } 2ab^2 \\
 2ab^2(64a^3 + 27b^3) & \text{Sum of cubes! Cube roots are } 4a \text{ and } 3b \\
 2ab^2(4a + 3b)(16a^2 - 12ab + 9b^2) & \text{Our Solution}
 \end{array}$$

6.5 Practice - Factoring Special Products

Factor each completely.

1) $r^2 - 16$

3) $v^2 - 25$

5) $p^2 - 4$

7) $9k^2 - 4$

9) $3x^2 - 27$

11) $16x^2 - 36$

13) $18a^2 - 50b^2$

15) $a^2 - 2a + 1$

17) $x^2 + 6x + 9$

19) $x^2 - 6x + 9$

21) $25p^2 - 10p + 1$

23) $25a^2 + 30ab + 9b^2$

25) $4a^2 - 20ab + 25b^2$

27) $8x^2 - 24xy + 18y^2$

29) $8 - m^3$

31) $x^3 - 64$

33) $216 - u^3$

35) $125a^3 - 64$

37) $64x^3 + 27y^3$

39) $54x^3 + 250y^3$

41) $a^4 - 81$

43) $16 - z^4$

45) $x^4 - y^4$

47) $m^4 - 81b^4$

2) $x^2 - 9$

4) $x^2 - 1$

6) $4v^2 - 1$

8) $9a^2 - 1$

10) $5n^2 - 20$

12) $125x^2 + 45y^2$

14) $4m^2 + 64n^2$

16) $k^2 + 4k + 4$

18) $n^2 - 8n + 16$

20) $k^2 - 4k + 4$

22) $x^2 + 2x + 1$

24) $x^2 + 8xy + 16y^2$

26) $18m^2 - 24mn + 8n^2$

28) $20x^2 + 20xy + 5y^2$

30) $x^3 + 64$

32) $x^3 + 8$

34) $125x^3 - 216$

36) $64x^3 - 27$

38) $32m^3 - 108n^3$

40) $375m^3 + 648n^3$

42) $x^4 - 256$

44) $n^4 - 1$

46) $16a^4 - b^4$

48) $81c^4 - 16d^4$

Factoring - Factoring Strategy

Objective: Identify and use the correct method to factor various polynomials.

With so many different tools used to factor, it is easy to get lost as to which tool to use when. Here we will attempt to organize all the different factoring types we have seen. A large part of deciding how to solve a problem is based on how many terms are in the problem. For all problem types we will always try to factor out the GCF first.

Factoring Strategy (GCF First!!!!)

- **2 terms:** sum or difference of squares or cubes:

$$a^2 - b^2 = (a + b)(a - b)$$

$$a^2 + b^2 = \text{Prime}$$

$$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$$

$$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$$

- **3 terms:** ac method, watch for perfect square!

$$a^2 + 2ab + b^2 = (a + b)^2$$

Multiply to ac and add to b

- **4 terms:** grouping

We will use the above strategy to factor each of the following examples. Here the emphasis will be on which strategy to use rather than the steps used in that method.

Example 312.

$$4x^2 + 56xy + 196y^2 \quad \text{GCF first, 4}$$

$$4(x^2 + 14xy + 49y^2) \quad \text{Three terms, try ac method, multiply to 49, add to 14}$$

7 and 7, perfect square!

$$4(x + 7y)^2 \quad \text{Our Solution}$$

Example 313.

$$\begin{array}{ll} 5x^2y + 15xy - 35x^2 - 105x & \text{GCF first, } 5x \\ 5x(xy + 3y - 7x - 21) & \text{Four terms, try grouping} \\ 5x[y(x + 3) - 7(x + 3)] & (x + 3) \text{ match!} \\ 5x(x + 3)(y - 7) & \text{Our Solution} \end{array}$$

Example 314.

$$\begin{array}{ll} 100x^2 - 400 & \text{GCF first, } 100 \\ 100(x^2 - 4) & \text{Two terms, difference of squares} \\ 100(x + 4)(x - 4) & \text{Our Solution} \end{array}$$

Example 315.

$$\begin{array}{ll} 108x^3y^2 - 39x^2y^2 + 3xy^2 & \text{GCF first, } 3xy^2 \\ 3xy^2(36x^2 - 13x + 1) & \text{Three terms, ac method, multiply to 36, add to } -13 \\ 3xy^2(36x^2 - 9x - 4x + 1) & -9 \text{ and } -4, \text{ split middle term} \\ 3xy^2[9x(4x - 1) - 1(4x - 1)] & \text{Factor by grouping} \\ 3xy^2(4x - 1)(9x - 1) & \text{Our Solution} \end{array}$$

World View Note: Variables originated in ancient Greece where Aristotle would use a single capital letter to represent a number.

Example 316.

$$\begin{array}{ll} 5 + 625y^3 & \text{GCF first, } 5 \\ 5(1 + 125y^3) & \text{Two terms, sum of cubes} \\ 5(1 + 5y)(1 - 5y + 25y^2) & \text{Our Solution} \end{array}$$

It is important to be comfortable and confident not just with using all the factoring methods, but decided on which method to use. This is why practice is very important!

6.6 Practice - Factoring Strategy

Factor each completely.

1) $24az - 18ah + 60yz - 45yh$

3) $5u^2 - 9uv + 4v^2$

5) $-2x^3 + 128y^3$

7) $5n^3 + 7n^2 - 6n$

9) $54u^3 - 16$

11) $n^2 - n$

13) $x^2 - 4xy + 3y^2$

15) $9x^2 - 25y^2$

17) $m^2 - 4n^2$

19) $36b^2c - 16xd - 24b^2d + 24xc$

21) $128 + 54x^3$

23) $2x^3 + 6x^2y - 20y^2x$

25) $n^3 + 7n^2 + 10n$

27) $27x^3 - 64$

29) $5x^2 + 2x$

31) $3k^3 - 27k^2 + 60k$

33) $mn - 12x + 3m - 4xn$

35) $16x^2 - 8xy + y^2$

37) $27m^2 - 48n^2$

39) $9x^3 + 21x^2y - 60y^2x$

41) $2m^2 + 6mn - 20n^2$

2) $2x^2 - 11x + 15$

4) $16x^2 + 48xy + 36y^2$

6) $20uv - 60u^3 - 5xv + 15xu^2$

8) $2x^3 + 5x^2y + 3y^2x$

10) $54 - 128x^3$

12) $5x^2 - 22x - 15$

14) $45u^2 - 150uv + 125v^2$

16) $x^3 - 27y^3$

18) $12ab - 18a + 6nb - 9n$

20) $3m^3 - 6m^2n - 24n^2m$

22) $64m^3 + 27n^3$

24) $3ac + 15ad^2 + x^2c + 5x^2d^2$

26) $64m^3 - n^3$

28) $16a^2 - 9b^2$

30) $2x^2 - 10x + 12$

32) $32x^2 - 18y^2$

34) $2k^2 + k - 10$

36) $v^2 + v$

38) $x^3 + 4x^2$

40) $9n^3 - 3n^2$

42) $2u^2v^2 - 11uv^3 + 15v^4$

Factoring - Solve by Factoring

Objective: Solve quadratic equation by factoring and using the zero product rule.

When solving linear equations such as $2x - 5 = 21$ we can solve for the variable directly by adding 5 and dividing by 2 to get 13. However, when we have x^2 (or a higher power of x) we cannot just isolate the variable as we did with the linear equations. One method that we can use to solve for the variable is known as the zero product rule

Zero Product Rule: If $ab = 0$ then either $a = 0$ or $b = 0$

The zero product rule tells us that if two factors are multiplied together and the answer is zero, then one of the factors must be zero. We can use this to help us solve factored polynomials as in the following example.

Example 317.

$$\begin{array}{ll}
 (2x - 3)(5x + 1) = 0 & \text{One factor must be zero} \\
 2x - 3 = 0 \text{ or } 5x + 1 = 0 & \text{Set each factor equal to zero} \\
 \begin{array}{r} +3+3 \\ \hline 2x=3 \end{array} \text{ or } \begin{array}{r} -1-1 \\ \hline 5x=-1 \end{array} & \text{Solve each equation} \\
 \begin{array}{r} \frac{2x}{2} = \frac{3}{2} \end{array} \text{ or } \begin{array}{r} \frac{5x}{5} = \frac{-1}{5} \end{array} & \\
 x = \frac{3}{2} \text{ or } x = \frac{-1}{5} & \text{Our Solution}
 \end{array}$$

For the zero product rule to work we must have factors to set equal to zero. This means if the problem is not already factored we will factor it first.

Example 318.

$$\begin{array}{ll}
 4x^2 + x - 3 = 0 & \text{Factor using the ac method, multiply to } -12, \text{ add to } 1 \\
 4x^2 - 3x + 4x - 3 = 0 & \text{The numbers are } -3 \text{ and } 4, \text{ split the middle term} \\
 x(4x - 3) + 1(4x - 3) = 0 & \text{Factor by grouping} \\
 (4x - 3)(x + 1) = 0 & \text{One factor must be zero} \\
 4x - 3 = 0 \text{ or } x + 1 = 0 & \text{Set each factor equal to zero}
 \end{array}$$

$$\begin{array}{rcl}
\frac{+3+3}{4x=3} & \frac{-1-1}{x=-1} & \text{Solve each equation} \\
\hline
\frac{4x=3}{4} & \text{or} & \frac{x=-1}{4} \\
x=\frac{3}{4} & \text{or} & -1 \quad \text{Our Solution}
\end{array}$$

Another important part of the zero product rule is that before we factor, the equation must equal zero. If it does not, we must move terms around so it does equal zero. Generally we like the x^2 term to be positive.

Example 319.

$$\begin{array}{rcl}
x^2 = 8x - 15 & & \text{Set equal to zero by moving terms to the left} \\
\frac{-8x+15}{x^2-8x+15=0} & \frac{-8x+15}{x^2-8x+15=0} & \text{Factor using the ac method, multiply to 15, add to } -8 \\
(x-5)(x-3)=0 & & \text{The numbers are } -5 \text{ and } -3 \\
x-5=0 \text{ or } x-3=0 & & \text{Set each factor equal to zero} \\
\frac{+5+5}{x=5} & \frac{+3+3}{x=3} & \text{Solve each equation} \\
x=5 \text{ or } x=3 & & \text{Our Solution}
\end{array}$$

Example 320.

$$\begin{array}{rcl}
(x-7)(x+3)=-9 & & \text{Not equal to zero, multiply first, use FOIL} \\
x^2-7x+3x-21=-9 & & \text{Combine like terms} \\
x^2-4x-21=-9 & & \text{Move } -9 \text{ to other side so equation equals zero} \\
\frac{+9}{x^2-4x-12=0} & \frac{+9}{x^2-4x-12=0} & \\
x^2-4x-12=0 & & \text{Factor using the ac method, multiply to } -12, \text{ add to } -4 \\
(x-6)(x+2)=0 & & \text{The numbers are } 6 \text{ and } -2 \\
x-6=0 \text{ or } x+2=0 & & \text{Set each factor equal to zero} \\
\frac{+6+6}{x=6} & \frac{-2-2}{x=-2} & \text{Solve each equation} \\
x=6 \text{ or } -2 & & \text{Our Solution}
\end{array}$$

Example 321.

$$\begin{array}{rcl}
3x^2+4x-5=7x^2+4x-14 & & \text{Set equal to zero by moving terms to the right} \\
\frac{-3x^2-4x+5}{0=4x^2-9} & \frac{-3x^2-4x+5}{0=4x^2-9} & \text{Factor using difference of squares}
\end{array}$$

$$\begin{array}{ll}
0 = (2x + 3)(2x - 3) & \text{One factor must be zero} \\
2x + 3 = 0 \text{ or } 2x - 3 = 0 & \text{Set each factor equal to zero} \\
\frac{-3-3}{2} \quad \frac{+3+3}{2} & \text{Solve each equation} \\
\frac{2x}{2} = \frac{-3}{2} \text{ or } \frac{2x}{2} = \frac{3}{2} & \\
x = \frac{-3}{2} \text{ or } \frac{3}{2} & \text{Our Solution}
\end{array}$$

Most problems with x^2 will have two unique solutions. However, it is possible to have only one solution as the next example illustrates.

Example 322.

$$\begin{array}{ll}
4x^2 = 12x - 9 & \text{Set equal to zero by moving terms to left} \\
\frac{-12x+9}{4x^2-12x+9=0} \quad \frac{-12x+9}{(2x-3)^2=0} & \text{Factor using the ac method, multiply to 36, add to } -12 \\
& \text{ } -6 \text{ and } -6, a \text{ perfect square!} \\
2x - 3 = 0 & \text{Set this factor equal to zero} \\
\frac{+3+3}{2x=3} & \text{Solve the equation} \\
\frac{2x}{2} = \frac{3}{2} & \\
x = \frac{3}{2} & \text{Our Solution}
\end{array}$$

As always it will be important to factor out the GCF first if we have one. This GCF is also a factor and must also be set equal to zero using the zero product rule. This may give us more than just two solution. The next few examples illustrate this.

Example 323.

$$\begin{array}{ll}
4x^2 = 8x & \text{Set equal to zero by moving the terms to left} \\
-8x - 8x & \text{Be careful, on the right side, they are not like terms!} \\
4x^2 - 8x = 0 & \text{Factor out the GCF of } 4x \\
4x(x - 2) = 0 & \text{One factor must be zero} \\
4x = 0 \text{ or } x - 2 = 0 & \text{Set each factor equal to zero} \\
\frac{4}{4} \quad \frac{4}{4} \quad \frac{+2+2}{4} & \text{Solve each equation} \\
x = 0 \text{ or } 2 & \text{Our Solution}
\end{array}$$

Example 324.

$2x^3 - 14x^2 + 24x = 0$	Factor out the GCF of $2x$
$2x(x^2 - 7x + 12) = 0$	Factor with ac method, multiply to 12, add to -7
$2x(x - 3)(x - 4) = 0$	The numbers are -3 and -4
$2x = 0$ or $x - 3 = 0$ or $x - 4 = 0$	Set each factor equal to zero
$\frac{2x}{2} = 0$ or $\frac{x-3}{2} = 0$ or $\frac{x-4}{2} = 0$	Solve each equation
$x = 0$ or 3 or 4	Our Solutions

Example 325.

$6x^2 + 21x - 27 = 0$	Factor out the GCF of 3
$3(2x^2 + 7x - 9) = 0$	Factor with ac method, multiply to -18 , add to 7
$3(2x^2 + 9x - 2x - 9) = 0$	The numbers are 9 and -2
$3[x(2x + 9) - 1(2x + 9)] = 0$	Factor by grouping
$3(2x + 9)(x - 1) = 0$	One factor must be zero
$3 = 0$ or $2x + 9 = 0$ or $x - 1 = 0$	Set each factor equal to zero
$3 \neq 0$ or $\frac{-9-9}{2} = 0$ or $\frac{-1+1}{2} = 0$	Solve each equation
$2x = -9$ or $x = 1$	
$x = -\frac{9}{2}$ or 1	Our Solution

In the previous example, the GCF did not have a variable in it. When we set this factor equal to zero we got a false statement. No solutions come from this factor. Often a student will skip setting the GCF factor equal to zero if there is no variables in the GCF.

Just as not all polynomials cannot factor, all equations cannot be solved by factoring. If an equation does not factor we will have to solve it using another method. These other methods are saved for another section.

World View Note: While factoring works great to solve problems with x^2 , Tartaglia, in 16th century Italy, developed a method to solve problems with x^3 . He kept his method a secret until another mathematician, Cardan, talked him out of his secret and published the results. To this day the formula is known as Cardan's Formula.

A question often asked is if it is possible to get rid of the square on the variable by taking the square root of both sides. While it is possible, there are a few properties of square roots that we have not covered yet and thus it is common to break a rule of roots that we are not aware of at this point. The short reason we want to avoid this for now is because taking a square root will only give us one of the two answers. When we talk about roots we will come back to problems like these and see how we can solve using square roots in a method called completing the square. For now, **never** take the square root of both sides!

6.7 Practice - Solve by Factoring

Solve each equation by factoring.

1) $(k - 7)(k + 2) = 0$

2) $(a + 4)(a - 3) = 0$

3) $(x - 1)(x + 4) = 0$

4) $(2x + 5)(x - 7) = 0$

5) $6x^2 - 150 = 0$

6) $p^2 + 4p - 32 = 0$

7) $2n^2 + 10n - 28 = 0$

8) $m^2 - m - 30 = 0$

9) $7x^2 + 26x + 15 = 0$

10) $40r^2 - 285r - 280 = 0$

11) $5n^2 - 9n - 2 = 0$

12) $2b^2 - 3b - 2 = 0$

13) $x^2 - 4x - 8 = -8$

14) $v^2 - 8v - 3 = -3$

15) $x^2 - 5x - 1 = -5$

16) $a^2 - 6a + 6 = -2$

17) $49p^2 + 371p - 163 = 5$

18) $7k^2 + 57k + 13 = 5$

19) $7x^2 + 17x - 20 = -8$

20) $4n^2 - 13n + 8 = 5$

21) $7r^2 + 84 = -49r$

22) $7m^2 - 224 = 28m$

23) $x^2 - 6x = 16$

24) $7n^2 - 28n = 0$

25) $3v^2 + 7v = 40$

26) $6b^2 = 5 + 7b$

27) $35x^2 + 120x = -45$

28) $9n^2 + 39n = -36$

29) $4k^2 + 18k - 23 = 6k - 7$

30) $a^2 + 7a - 9 = -3 + 6a$

31) $9x^2 - 46 + 7x = 7x + 8x^2 + 3$

32) $x^2 + 10x + 30 = 6$

33) $2m^2 + 19m + 40 = -2m$

34) $5n^2 + 41n + 40 = -2$

35) $40p^2 + 183p - 168 = p + 5p^2$

36) $24x^2 + 11x - 80 = 3x$