Part 1 - The Definition

Exponential notation expresses repeated multiplication.

Definition: We define 2^7 to denote the factor 2 multiplied by itself repeatedly 7 times, such as

$$\underbrace{2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2}_{7\text{ factors}} = 2^7$$

The new operation defined is called **exponentiation.** The factor (in this case 2) is called the **base**. The number written above the base, in smaller font (in this case, 7) is called the **exponent**.

Since the definition does not elegantly fit the case when the exponent is one, we also define 5^1 to be 5. One factor, so technically, no multiplication.

When we enlarge our mathematical notation by the inclusion of exponential expressions, a few things might become problematic. For example, is there a difference between -3^2 and $(-3)^2$?

Recall that a negative sign in front of anything can be interpreted as '*the opposite of*', which is the same as mutliplication by -1. We can interpret -3 as $-1 \cdot 3$, and so we can re-interpret the original question from comparing -3^2 and $(-3)^2$ to a question comparing $-1 \cdot 3^2$ and $(-1 \cdot 3)^2$. The rest is really just an order of operations problem.

Recall that in our order of operations agreement, [P / E / M D / A S], exponentiation superseeds multiplication. So, when presented by multiplication and exponentiation, we first execute the exponentiation and then the multiplication. And so, if there is no parentheses, we have

$-1 \cdot 3^2$	perform exponentiation
$-1 \cdot 9$	perform multiplication
-9	

And if we have a parentheses, that serves to overwrite the usual order of operations:

$(-1 \cdot 3)^2$	whatever is in the parentheses first
$(-3)^2$	square the number -3
9	

The difference between -3^2 and $(-3)^2$ is truly an order of operations thing: we are talking about taking the opposite and squaring, but in different orders.

-3^2 is square 3 and then take the result's opposite —- or the opposite of the square of 3

$(-3)^2$ is take the opposite of 3 and then square —— or the square of the opposite of 3

In algebra, it is important to correctly read notation. Confusing -3^2 and $(-3)^2$ is an error that commonly occurs and messes up computations.

> Caution! In the expression $(-3)^2$, the base of the exponentiation is -3. In the expression -3^2, the base of the exponentiation is 3.

Example 1. Simplify each of the given expressions.

a) -2^4 b) $(-2)^4$ c) -1^3 d) $(-1)^3$ e) $-(-2)^2$ f) $-\left(-2^2\right)$

Solution: a) The base of the exponentiation is 2.

$$-2^4 = -1 \cdot 2^4 = -1 \cdot (2 \cdot 2 \cdot 2 \cdot 2) = -1 \cdot 16 = \boxed{-16}$$

-2^4 can be read as the opposite of 2^4.

b) The base of the exponentiation is -2.

$$(-2)^4 = (-1 \cdot 2)^4 = (-1 \cdot 2)(-1 \cdot 2)(-1 \cdot 2)(-1 \cdot 2) = (-2)(-2)(-2)(-2) = \boxed{16}$$

$(-2)^4$ can be read as the fourth power of -2.

c) The base of the exponentiation is 1.

$$-1^3 = -1 \cdot 1^3 = -1 \cdot 1 \cdot 1 \cdot 1 = \boxed{-1}$$

d) The base of the exponentiation is -1.

$$(-1)^3 = (-1)(-1)(-1) = \boxed{-1}$$

e) The base of the exponentiation is -2.

$$-(-2)^2 = -((-2)(-2)) = -(4) = \boxed{-4}$$

f) Careful! The base of the exponentiation is 2. This is NOT squaring -2. This is squaring 2 and then taking the opposite of the result twice.

$$-\left(-2^2\right) = -(-1 \cdot 2 \cdot 2) = -(-4) = \boxed{4}$$

Discussion: Explain why in the expression $-(-5)^2$, the two negatives do not cancel out to a positive.

Part 2 - Rules of Exponents

When mathematicians agreed to define 3^5 as $3 \cdot 3 \cdot 3 \cdot 3 \cdot 3$, that was a free choice. They could have gone with other definitions. Once this definition exists, however, certain properties are automatically true, and we have no other option but to recognize them as true. The following statements are straightforward consequences of the definition - and the mathematics we already have.

Consider the expression $2^3 \cdot 2^4$. If we re-write the expression using the definition of exponents, we quickly get that

$$2^3 \cdot 2^4 \overset{\text{def}}{=} (2 \cdot 2 \cdot 2) \cdot (2 \cdot 2 \cdot 2 \cdot 2) \underset{\text{associative}}{\overset{\text{mult is}}{=}} 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \overset{\text{def}}{=} 2^7$$

The computation above illustrates why the following theorem is true.

> **Theorem 1.** If a is any number and m and n are any two positive integers, then
> $$a^n \cdot a^m = a^{n+m}$$

We can see that this rule follows from the definition of exponents and from the fact that multiplication is associative.

Consider now the expression $\dfrac{2^5}{2^3}$. If we re-write the expression using the definition of exponents, we quickly get that

$$\frac{2^5}{2^3} \overset{\text{def}}{=} \frac{2\cdot2\cdot2\cdot2\cdot2}{2\cdot2\cdot2} \overset{\text{cancellation}}{=} \frac{\cancel{2}\cdot\cancel{2}\cdot\cancel{2}\cdot2\cdot2}{\cancel{2}\cdot\cancel{2}\cdot\cancel{2}} \overset{\text{def}}{=} \frac{2^2}{1} = 2^2$$

> **Theorem 2.** If a is any number and m and n are any two positive integers, then
> $$\frac{a^n}{a^m} = a^{n-m}$$

As our computation shows, this property is a consequence of the definition of exponentials and our rules of cancellation.

Consider now $\left(2^3\right)^5$. If we re-write the expression using the definition of exponents, we quickly get that

$$\left(2^3\right)^5 \overset{\text{def}}{=} \left(2^3\right)\cdot\left(2^3\right)\cdot\left(2^3\right)\cdot\left(2^3\right)\cdot\left(2^3\right) \overset{\text{def}}{=} (2\cdot2\cdot2)\cdot(2\cdot2\cdot2)\cdot(2\cdot2\cdot2)\cdot(2\cdot2\cdot2)\cdot(2\cdot2\cdot2)$$
$$\underset{\text{associative}}{\overset{\text{mult is}}{=}} 2\cdot2\cdot2\cdot2\cdot2\cdot2\cdot2\cdot2\cdot2\cdot2\cdot2\cdot2\cdot2\cdot2\cdot2 \overset{\text{def}}{=} 2^{15}$$

Clearly, we have five groups of three two-factors.

> **Theorem 3.** If a is any number and m and n are any two positive integers, then
> $$(a^n)^m = a^{nm}$$

As our computation shows, this property is a consequence of the definition of exponentials and the fact that multiplication is associative.

Consider now $(2\cdot3)^4$. If we re-write the expression using the definition of exponents, we quickly get that

$$(2\cdot3)^4 \overset{\text{def}}{=} (2\cdot3)\cdot(2\cdot3)\cdot(2\cdot3)\cdot(2\cdot3) \underset{\text{associative}}{\overset{\text{mult is}}{=}} 2\cdot3\cdot2\cdot3\cdot2\cdot3\cdot2\cdot3$$
$$\underset{\text{commutative}}{\overset{\text{mult is}}{=}} 2\cdot2\cdot2\cdot2\cdot3\cdot3\cdot3\cdot3 \underset{\text{associative}}{\overset{\text{mult is}}{=}} (2\cdot2\cdot2\cdot2)\cdot(3\cdot3\cdot3\cdot3) \overset{\text{def}}{=} 2^4\cdot3^4$$

> **Theorem 4.** If a and b are any numbers and n is any positive integer, then
> $$(ab)^n = a^n\cdot b^n$$

As our computation shows, this property is a consequence of the definition of exponentials and the fact that multiplication is commutative and associative.

Caution! Exponentiation denotes repeated multiplication, so it is a fundamentally multiplicative concept. Exponents will exhibit nice behvior with respect to multiplication and division, but NOT with respect to addition and subtraction. Similar-looking statements fail to be true if addition or subtraction is involved. For example, $(2 \cdot 5)^2 = 2^2 \cdot 5^2$, but $(2+5)^2 \neq 2^2 + 5^2$.

Caution! Another common mistake is to confuse $(ab)^n$ with ab^n. (It is an order of operations thing.) The base of exponentiation is ab in $(ab)^n$ but only b in ab^n.

$$ab^n = a \cdot \underbrace{b \cdot b \cdot \ldots \cdot b}_{n \text{ times}} \quad \text{and} \quad (ab)^n = \underbrace{(ab) \cdot (ab) \cdot \ldots \cdot (ab)}_{n \text{ times}}$$

Consider now the expression $\left(\frac{3}{4}\right)^5$. If we re-write the expression using the definition of exponents, we quickly get that

$$\left(\frac{3}{4}\right)^5 \stackrel{\text{def}}{=} \left(\frac{3}{4}\right) \cdot \left(\frac{3}{4}\right) \cdot \left(\frac{3}{4}\right) \cdot \left(\frac{3}{4}\right) \cdot \left(\frac{3}{4}\right) \underset{\text{fractions}}{\stackrel{\text{rules of multipying}}{=}} \frac{3 \cdot 3 \cdot 3 \cdot 3 \cdot 3}{4 \cdot 4 \cdot 4 \cdot 4 \cdot 4} \stackrel{\text{def}}{=} \frac{3^5}{4^5}$$

Theorem 5. If a and b are any numbers and n is any positive integer, then

$$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$$

As our computation shows, this property is a consequence of the definition of exponentials and the rule of how we multiply fractions.

Caution! Exponentiation denotes repeated multiplication, so it is a fundamentally multiplicative concept. Exponents will exhibit nice behavior with respect to multiplication and division, but NOT with respect to addition and subtraction. Similar-looking statements fail to be true if addition or subtraction is involved. For example, $\left(\frac{2}{5}\right)^2 = \frac{2^2}{5^2}$, but $(5-2)^2 \neq 5^2 - 2^2$.

To summarize what just happened: once we defined exponentiation as repeated multiplication, certain properties immediately followed from the definition. These properties are as follows.

Theorem: If a, b are any numbers and m, n are any positive integers, then

1. $a^n \cdot a^m = a^{n+m}$
2. $\dfrac{a^n}{a^m} = a^{n-m}$
3. $(a^n)^m = a^{nm}$
4. $(ab)^n = a^n b^n$
5. $\left(\dfrac{a}{b}\right)^n = \dfrac{a^n}{b^n}$

Example 2. Simplify each of the following expressions.

a) $x^5 \cdot x^3$ b) $\left(x^5\right)^3$ c) $\left(2x^5\right)^3$ d) $\dfrac{(2x)^5}{2x^3}$ e) $\left(-x^2\right)^3$ f) $\left(-x^3\right)^2$

Solution: a) In case of $x^5 \cdot x^3$, we can apply the definition or our first rule. Either way, we will end up adding the exponents.

$$x^5 \cdot x^3 = \boxed{x^8}$$

b) In case of $\left(x^5\right)^3$, we have repeated exponentiation. Applying the definition, we see three groups of factors, each group with five factors in it. So, we multiply the exponents.

$$\left(x^5\right)^3 = \boxed{x^{15}}$$

c) In case of $\left(2x^5\right)^3$, we will have to apply several rules. First, when a product is exponentiated, we exponentiate each factor, i.e. $(ab)^n = a^n b^n$.

$$\left(2x^5\right)^3 = 2^3 \cdot \left(x^5\right)^3 = 8 \cdot x^{15} = \boxed{8x^{15}}$$

d) In the expression $\dfrac{(2x)^5}{2x^3}$, the number 2 is part of the base in the exponentiation in the numerator, but not in the exponentiation in the denominator. Once we got rid of the parentheses in the numerator, we subtract the exponents corresponding to cancellation.

$$\frac{(2x)^5}{2x^3} = \frac{2^5 \cdot x^5}{2x^3} = \frac{32x^5}{2x^3} = 16x^{5-3} = \boxed{16x^2}.$$

e) In the expression $\left(-x^2\right)^3$, the leading negative sign brings some algebraic complications. One way to handle this, either mentally or also in writing, to interpret $-x^2$ as the opposite of x^2, or $-1 \cdot x^2$. Notice that the -1 is not getting raised to second power, only to the third power.

$$\left(-x^2\right)^3 = \left(-1 \cdot x^2\right)^3 = (-1)^3 \left(x^2\right)^3 = -1 \cdot x^6 = \boxed{-x^6}.$$

f) This expression is very similar to the previous one. Here the negative sign gets squared only but not cubed.

$$\left(-x^3\right)^2 = \left(-1 \cdot x^3\right)^2 = (-1)^2 \left(x^3\right)^2 = 1 \cdot x^6 = \boxed{x^6}.$$

Example 3. Simplify the expression $\dfrac{(-2a)^4 \left(-ab^4\right)^3 ab^2}{\left(-2ab^2\right)^3 ba^2}$. Assume that a and b represent non-zero numbers.

Solution: There are several ways to solve this problem. We will take our time and apply one rule at the time. We will start with the rule $(ab)^n = a^n b^n$. Also, standalone negative signs will be replaced with a -1 multiplyer.

$$\frac{(-2a)^4 \left(-ab^4\right)^3 ab^2}{\left(-2ab^2\right)^3 ba^2} = \frac{(-2a)^4 \left(-1ab^4\right)^3 ab^2}{\left(-2ab^2\right)^3 ba^2} = \frac{(-2)^4 a^4 (-1)^3 a^3 \left(b^4\right)^3 ab^2}{(-2)^3 a^3 \left(b^2\right)^3 ba^2}$$

Next, we perform the exponentiation on the numbers and bring them forward, re-ordered the variables alphabetically, and simplify expressions contining repeated exponentiation using the rule $(a^n)^m = a^{nm}$.

$$\frac{(-2)^4 a^4 (-1)^3 a^3 \left(b^4\right)^3 ab^2}{(-2)^3 a^3 \left(b^2\right)^3 ba^2} = \frac{16a^4 (-1) a^3 b^{12} ab^2}{-8a^3 b^6 ba^2} = \frac{-16a^4 a^3 ab^{12} b^2}{-8a^3 a^2 b^6 b}$$

Next, we apply our first rule, $a^n a^m = a^{n+m}$ in both numerator and denominator.

$$\frac{-16a^4 a^3 ab^{12} b^2}{-8a^3 a^2 b^6 b} = \frac{-16a^{4+3+1} b^{12+2}}{-8a^{3+2} b^{6+1}} = \frac{-16a^8 b^{14}}{-8a^5 b^7}$$

At this point, both numerator and denominator are completely simplified. We can tell because both expressions have no repetition of sign, number, or any of the variables. What is left for us to do, is to consolidate numerator and denominator. We will simplify (or perform the division) between -16 and -8, and perform cancellation between the variables, using the rule $\dfrac{a^n}{a^m} = a^{n-m}$.

$$\frac{-16a^8b^{14}}{-8a^5b^7} = \frac{-2a^{8-5}b^{14-7}}{1} = 2a^3b^7$$

So the simplified expression is $\boxed{2a^3b^7}$.

Discussion: Why was it necessary in the previous example to assume that a and b represent non-negative numbers?

Explain why this step is incorrect: $2 \cdot 5^x = 10^x$.

Example 4. Consider the expression $3^x \cdot 3^x$.

a) Simplify the expression using our first rule, $a^n \cdot a^m = a^{n+m}$ only.

b) Simplify the expression using our fourth rule, $(ab)^n = a^n b^n$ only.

c) Is there a rule of exponentiation that can be used to verify that the results from part a) and part b) are the same?

Solution: a) The base is the same, so

$$3^x \cdot 3^x = 3^{x+x} = \boxed{3^{2x}}.$$

b) Now we will use the fact that the exponents of factors are the same, and so we will re-write $a^n \cdot b^n$ as $(ab)^n$.

$$3^x \cdot 3^x = (3 \cdot 3)^x = \boxed{9^x}$$

c) The expressions 3^{2x} and 9^x are the same. We can use our third rule, $(a^n)^m = a^{nm}$.

$$9^x = \left(3^2\right)^x = 3^{2 \cdot x} = 3^{2x}.$$

Example 5. Find the prime-factorization of 24^{100}.

Solution: If we tried to enter 24^{100} in our calculator, we will probably find that the number is too great for it to handle. So it would be futile to compute the gigantic number and start the prime-factorization from scratch. Instead, we will use the prime-factorization of 24 and rules of exponents. The prime-factorization of $24 = 2^3 \cdot 3$.

$$24^{100} = \left(2^3 \cdot 3\right)^{100} = \left(2^3\right)^{100} \cdot 3^{100} = 2^{300} \cdot 3^{100}.$$

So the prime-factorization of 24^{100} is $\boxed{2^{300} \cdot 3^{100}}$.

Part 3 - Scientific Notation

In physics, we use units that are internationally established by scientists. The abbreviation SI stands for (Système international d'unités). In natural sciences such as chemistry and physics, we often have to communicate numbers so large that it is uncomfortable for both of our imagination and even for our notation. Consider for example, the radius of the sun. The SI unit of measuring length is meters.

The radius of the sun is 695 700 000 meters. The distance between our planet Earth and the Sun is approximately 149 600 000 000 meters. The basic particles that make up material are so tiny that a handful of some material includes a very large number of particles. In chemistry, the basic measurement of how how much material we have (i.e. how many particles) is 1 mole. One mole of a substance contains approximately 602 214 076 000 000 000 000 000 particles.

Scientific notation was developed to handle such uncomfortably large numbers, using properties of exponentiation. Scientific notation expresses a single number as a product, where the first number (kind of, sort of) expresses the number, and the second part expresses the number of zeroes to place after the number. Naturally, the definition is going to be a bit more rigorous.

Example 6. Re-write $4 \cdot 10^7$ using regular notation.

Solution: If we look at ten-powers, we will notice some very nice properties.

$10^1 = 10$ and multiplying an integer by 10 results in adding a zero as its last digit. $4 \cdot 10^1 = 40$.

$10^2 = 100$ and multiplying an integer by 100 results in adding two zeroes as its last digits. $4 \cdot 10^2 = 400$.

$10^3 = 1000$ and multiplying an integer by 1000 results in adding three zeroes as its last digits. $4 \cdot 10^3 = 4000$.

$10^4 = 10\,000$ and multiplying an integer by 10 000 results in adding four zeroes as its last digits. $4 \cdot 10^4 = 40\,000$.

This is indeed a very nice pattern. The exponent on 10 is the same as the number of zeroes to be added at the end. Therefore, we can interpret $4 \cdot 10^7$ as placing 7 zeroes after the digit 4.

$$4 \cdot 10^7 = \boxed{40\,000\,000}.$$

Example 7. Re-write $3.215 \cdot 10^{12}$ using regular notation.

Solution: When we are dealing with decimals, a multiplication by a $10-$power means moving the decimal point.

$3.215 \cdot 10 = 32.15$, $3.215 \cdot 100 = 321.5$, $3.215 \cdot 1000 = 3215$, and $3.125 \cdot 10000 = 31250$

Once we reached the end of the decimal, i.e. multiplied it by a ten-power large enough to create an integer, multiplication by the remaining $10-$powers is again a matter of placing zeroes at the end.

$$3.215 \cdot 10^{12} = 3.215 \cdot 10^{3+9} = 3.215 \cdot 10^3 \cdot 10^9 = 3215 \cdot 10^9 = \boxed{3215\,000\,000\,000}.$$

Definition: We can write numbers in scientific notation. This means to write a number as a product of two numbers. The first number is between 1 and 10 (can be 1 but must be less than 10), and the second number is a $10-$power. For example, the scientific notation for 428 600 000 000 is 4.286×10^{11}.

Please note that scientific notation uses the cross notation for multiplication. We will break with tradition and simply use the dot notation for scientific notation. Part of the problem with the cross notation is that in hand-written computations it looks too much like the letter x.

Example 8. Re-write 602 200 000 000 000 000 000 000 using scientific notation.

Solution: Let us first count the trailing zeroes at the end. There are six groups of three zeroes and then two more, so that is 20. So now we can re-write this giant number as $6022 \cdot 10^{20}$. But remember that the first factor in scientific notation must be between 1 and 10. So we will extract more ten-powers by moving the decimal point. When in doubt, check with the calculator. $6022 = 602.2 \cdot 10 = 60.22 \cdot 100 = 6.022 \cdot 1000$. Thus our number is

$$602\,200\,000\,000\,000\,000\,000\,000 = 6022 \cdot 10^{20} = (6.022 \cdot 1000) \cdot 10^{20} = 6.022 \cdot 10^{3+20} = \boxed{6.022 \cdot 10^{23}}$$

This number is famous in chemistry, it is called Avogadro's number. 1 mole of a substance contains $6.022 \cdot 10^{23}$ particles. Mole is the SI unit for the amount of a substance.

Example 9. Suppose that two numbers, A and B are given in scientfici notation as follows. $A = 3 \cdot 10^8$ and $B = 6 \cdot 10^3$. Compute each of the following. Present your answer in scientific notation.

a) $A + B$ b) $A - B$ c) AB d) $\dfrac{A}{B}$

Solution: a) Surprisingly, $A + B$ will look a whole lot like A. Scientific notation will serve us very well in multiplications and divisions, but we must be careful when adding or subtracting. To see what happens, we will return to regular notation. Notice also that there is no rule for how to add exponential expressions in the rules we just learned.

$A = 3 \cdot 10^8 = 300\,000\,000$ and $B = 6 \cdot 10^3 = 6000$. So $A + B = 300\,000\,000 + 6000 = 300\,006\,000$.

This can be written in scientific notation as $3.00006 \cdot 10^3$ or simply $3 \cdot 10^8$. Basically, A is so much larger than B, that the addition of B is almost negligible compared to the size of A. (Imagine that we added 1 inch to 1 mile or a penny to a million dollars) Indeed, in science, there will be agreements about the precision of results, and in most agreements, $3 \cdot 10^8 + 6 \cdot 10^3$ is simply $3 \cdot 10^8$, as we round 3.00006 down to 3. So the answer is $\boxed{300\,006\,000}$ or $\boxed{3.00006 \cdot 10^3 \approx 3 \cdot 10^8}$.

b) Very similarly, $A - B = 300\,000\,000 - 6000 = \boxed{299\,994\,000}$. Using scientific notation, the result is $\boxed{2.99994 \cdot 10^8 \approx 3 \cdot 10^8}$. We can imagine that we subtracted an inch from a mile or a penny from a million dollars.

c) Scientific notation will be awesome for mutiplication and division! Consider $AB = \left(3 \cdot 10^8\right)\left(6 \cdot 10^3\right)$. We will group the first factors and the ten-powers. $3 \cdot 6 = 18$ and $10^8 \cdot 10^3 = 10^{3+8} = 10^{11}$.

$AB = \left(3 \cdot 10^8\right)\left(6 \cdot 10^3\right) = (3 \cdot 6) \cdot \left(10^8 \cdot 10^3\right) = 18 \cdot 10^{11}$

This is not in scientific notation. Recall that in scientific notation, the first factor must be less than 10. So we have to re-write 18 as $1.8 \cdot 10$.

$AB = 18 \cdot 10^{11} = (1.8 \cdot 10) \cdot 10^{11} = 1.8 \cdot 10^{12}$. So the answer is $\boxed{1.8 \cdot 10^{12} = 1\,800\,000\,000\,000}$.

d) In order to help the division, we will re-write A from $3 \cdot 10^8$ to $30 \cdot 10^7$. Then the rules of exponents will work with the notation very nicely.

$$\frac{A}{B} = \frac{3 \cdot 10^8}{6 \cdot 10^3} = \frac{30 \cdot 10^7}{6 \cdot 10^3}$$

We simply divide 30 by 6. As for the ten-powers, then cancellation can be expressed via the rule $\dfrac{a^n}{a^m} = a^{n-m}$.

$$\frac{A}{B} = \frac{30 \cdot 10^7}{6 \cdot 10^3} = \frac{5 \cdot 10^{7-3}}{1} = \boxed{5 \cdot 10^4 \text{ or } 50\,000}.$$

Sample Problems

1. Simplify each of the following.

a) $\left(2x^5\right)\left(x^4\right)$ c) $\left(2x^5\right)^4$ e) $-2a^3\left(-2a^4\right)^2$ g) $\dfrac{\left(-2x^5\right)^2 y^3}{2x^3y^2}$

b) $\left(2x\right)^5\left(x^4\right)$ d) $\left(-xy\right)^2\left(-xy^2\right)^3$ f) $2a^3\left(-2ab^2\right)^3 ab^2$ h) $\dfrac{\left(2ab\right)^3\left(-3a^2b\right)^2}{-a\left(6ab^2\right)^2}$

2. Write each of the following expressions in terms of a fixed number or a single exponential expression.

a) $\dfrac{3^{2x+1}}{9^{x-1}}$ b) $\dfrac{\left(8^{b-2}\right)\left(2^{b+1}\right)}{4^{2b-3}}$ c) $5^{2x-1}\cdot 25^{3-x}$

3. Let us denote 3^{100} by M. Express each of the following in terms of M.

a) 3^{101} b) $3^{100}-2\cdot 3^{101}+3^{102}$ c) 3^{99} d) 9^{100}

4. Find the prime-factorization for each of the following numbers.

a) 10^{2018} b) 18^{1000} c) 360^{50} d) $10\cdot 9\cdot 8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1$

5. Re-write each of the given numbers using scientific notation.

a) 3800 000 000 b) 6250 000 000 000

6. Suppose that $A=5\cdot 10^{18}$, and $B=8\cdot 10^{7}$. Compute each of the following. Present your answer using scientific notation.

a) AB b) A^2 c) $2A$ d) $\dfrac{4A}{B}$

Practice Problems

1. Simplify each of the following.

   a) -3^2 c) $\left(-2a^3\right)^4$ e) $\left(2a^2b\right)^3$ g) $(2a)^2\,b^3$ i) $\left(3p^2q^5\right)\left(2pq^3\right)$

   b) $(-3)^2$ d) $\left(-2a^4\right)^3$ f) $\left((2a)^2\,b\right)^3$ h) $\dfrac{m^4m^5}{m^3}$ j) $\dfrac{\left(a^2\right)^6 a^3}{\left(-a^3\right)^2}$

   k) $\dfrac{\left(-5ts^3t\right)^3\left(4s^2t\right)^2}{\left(10st^3\right)^2}$ m) $\dfrac{\left(3ab^2\right)^2\left(-2a^3b\right)^4}{(-2ab)^3}$ o) $\left(\dfrac{6a^3b^5}{-3ab^2}\right)^2\left(\dfrac{12ab^3}{-6b^2}\right)^3$

   l) $\left(-2xy^3\right)^2 xy^5x^2$ n) $\dfrac{\left(-2x^2y^3\right)^4 xy^3\left(2x^2y\right)^2}{(2x)^2\,y^9\left(2x^2y\right)^4}$

2. Write each of the following expressions in terms of a fixed number or a single exponential expression.

   a) $\dfrac{2^{2x-1}}{4^{x-2}}$ b) $\dfrac{100^{x+1}}{2^{2x+1}\cdot 5^{x-1}}$ c) $\dfrac{9^x\cdot 4^{x+2}}{6^{2x-1}}$

3. Let P denote 5^{2015}. Express each of the following in terms of P.

   a) 5^{2016} b) 5^{2017} c) 5^{2014} d) 25^{2015} e) $5^{2015}-3\cdot 5^{2016}+5^{2017}$

4. Find the prime-factorization for each of the given numbers.

   a) 20^{100} b) 18^{2000} c) 120^{120} d) $(5\cdot 4\cdot 3\cdot 2\cdot 1)^3$

5. Re-write each of the given numbers in scientific notation.

   a) 21 000 000 000 b) 300 000 000 000 c) 325 000 000

6. Suppose that $X = 3\cdot 10^{10}$, and $Y = 6\cdot 10^4$. Compute each of the following. Present your answer using scientific notation.

   a) X^2 b) XY c) $\dfrac{X}{Y}$ d) $\dfrac{3X}{Y^2}$

Answers

Discussion

1. Two minus signs still cancel out each other, it is just that there are really three negative signs in the expression $-(-5)^2 = -(-5)(-5)$.

2. a and b appear in the denominator and division by zero is not allowed.

3. $2 \cdot 5^x = 2 \cdot \underbrace{5 \cdot 5 \cdot \ldots \cdot 5}_{x \text{ times}}$ Imagine that x is very large. Then we have lots of 5-factors but just one 2-factor. In order to be able to simplify to 10^x, we would need to see $2^x \cdot 5^x$.

Sample Problems

1. a) $2x^9$ b) $32x^9$ c) $16x^{20}$ d) $-x^5y^8$ e) $-8a^{11}$ f) $-16a^7b^8$ g) $2x^7y$ h) $-2a^4b$

2. a) 27 b) 2 c) 3125 3. a) $3M$ b) $4M$ c) $\dfrac{M}{3}$ d) M^2

4. a) $2^{2018} \cdot 5^{2018}$ b) $2^{1000} \cdot 3^{2000}$ c) $2^{150} \cdot 3^{100} \cdot 5^{50}$ d) $2^8 \cdot 3^4 \cdot 5^2 \cdot 7$

5. a) $3.8 \cdot 10^9$ b) $6.25 \cdot 10^{12}$ 6. a) $4 \cdot 10^{26}$ b) $2.5 \cdot 10^{37}$ c) $1 \cdot 10^{19}$ d) $2.5 \cdot 10^{11}$

6. a) $9 \cdot 10^{20}$ b) $1.8 \cdot 10^{15}$ c) $5 \cdot 10^5$ d) $2.5 \cdot 10$

Practice Problems

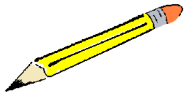
1. a) -9 b) 9 c) $16a^{12}$ d) $-8a^{12}$ e) $8a^6b^3$ f) $64a^6b^3$ g) $4a^2b^3$ h) m^6 i) $6p^3q^8$ j) a^9 k) $-20s^{11}t^2$ l) $4x^5y^{11}$ m) $-18a^{11}b^5$ n) x^3y^4 o) $-32a^7b^9$ 2. a) 8 b) $250 \cdot 5^x$ c) 96

3. a) $5P$ b) $25P$ c) $\dfrac{P}{5}$ d) P^2 e) $11P$

4. a) $2^{200}5^{100}$ b) $2^{2000} \cdot 3^{4000}$ c) $2^{360} \cdot 3^{120} \cdot 5^{120}$ d) $2^9 \cdot 3^3 \cdot 5^3$

5. a) $2.1 \cdot 10^{10}$ b) $3 \cdot 10^{14}$ c) $3.25 \cdot 10^8$

6. a) $9 \cdot 10^{20}$ b) $1.8 \cdot 10^{15}$ c) $5 \cdot 10^5$ d) $2.5 \cdot 10$



Sample Problems - Solutions

Let us recall the rules of exponents.

$$\begin{array}{lrcl} 1) & a^n \cdot a^m & = & a^{n+m} \\ 2) & \dfrac{a^n}{a^m} & = & a^{n-m} \\ 3) & (a^n)^m & = & a^{nm} \\ 4) & (ab)^n & = & a^n b^n \\ 5) & \left(\dfrac{a}{b}\right)^n & = & \dfrac{a^n}{b^n} \end{array}$$

1. Simplify each of the following.

a) $\left(2x^5\right)\left(x^4\right)$

Solution: $\left(2x^5\right)\left(x^4\right) = 2x^5x^4 = 2x^{5+4} = \boxed{2x^9}$ by rule 1.

b) $(2x)^5\left(x^4\right)$

Solution:

$$\begin{array}{rcll} (2x)^5\left(x^4\right) & = & 2^5x^5x^4 & \text{by rule 4} \\ & = & 32x^{5+4} & \text{by rule 1} \\ & = & \boxed{32x^9} & \end{array}$$

c) $\left(2x^5\right)^4$

Solution:

$$\begin{array}{rcll} \left(2x^5\right)^4 & = & 2^4\left(x^5\right)^4 & \text{by rule 4} \\ & = & \boxed{16x^{20}} & \text{by rule 3} \end{array}$$

d) $(-xy)^2\left(-xy^2\right)^3$

Solution:

$$\begin{array}{rcll} (-xy)^2\left(-xy^2\right)^3 & = & (-1xy)^2\left(-1xy^2\right)^3 & \text{the 1s will help with signs} \\ & = & (-1)^2x^2y^2(-1)^3x^3\left(y^2\right)^3 & \text{by rule 4} \\ & = & 1 \cdot x^2y^2(-1)x^3y^6 & \text{by rule 3} \\ & = & 1(-1)x^2x^3y^2y^6 & \text{multiplication is commutative} \\ & = & -1\,x^{2+3}y^{2+6} & \text{by rule 1} \\ & = & \boxed{-x^5y^8} & \end{array}$$

e) $-2a^3\left(-2a^4\right)^2$

Solution:

$$\begin{aligned} -2a^3\left(-2a^4\right)^2 &= -2a^3(-2)^2\left(a^4\right)^2 && \text{rule 4}\\ &= -2a^3(4)\,a^8 && \text{rule 3}\\ &= -2(4)\,a^3a^8 && \text{multiplication is commutative}\\ &= -8a^{3+8} = \boxed{-8a^{11}} && \text{rule 1} \end{aligned}$$

f) $2a^3\left(-2ab^2\right)^3ab^2$

Solution:

$$\begin{aligned} 2a^3\left(-2ab^2\right)^3ab^2 &= 2a^3(-2)^3a^3\left(b^2\right)^3ab^2 && \text{rule 4}\\ &= 2a^3(-8)\,a^3b^6ab^2 && \text{rule 3}\\ &= 2(-8)\,a^3a^3ab^6b^2 && \text{multiplication is commutative}\\ &= -16a^{3+3+1}b^{6+2} = \boxed{-16a^7b^8} && \text{rule 1} \end{aligned}$$

g) $\dfrac{\left(-2x^5\right)^2y^3}{2x^3y^2}$

Solution:

$$\begin{aligned} \frac{\left(-2x^5\right)^2y^3}{2x^3y^2} &= \frac{(-2)^2\left(x^5\right)^2y^3}{2x^3y^2} && \text{rule 4}\\ &= \frac{4x^{10}y^3}{2x^3y^2} && \text{rule 3}\\ &= \frac{4x^{10-3}y^{3-2}}{2} && \text{rule 2}\\ &= \frac{4x^7y^1}{2} = \boxed{2x^7y} \end{aligned}$$

h) $\dfrac{(2ab)^3\left(-3a^2b\right)^2}{-a\left(6ab^2\right)^2}$

Solution:

$$\begin{aligned} \frac{(2ab)^3\left(-3a^2b\right)^2}{-a\left(6ab^2\right)^2} &= \frac{(2ab)^3\left(-3a^2b\right)^2}{-1a\left(6ab^2\right)^2} && \text{the 1 will help with signs}\\ &= \frac{2^3a^3b^3(-3)^2\left(a^2\right)^2b^2}{-1\cdot a\cdot 6^2\cdot a^2\left(b^2\right)^2} && \text{by rule 4}\\ &= \frac{8a^3b^3\cdot 9\cdot a^4b^2}{-1\cdot a\cdot 36\cdot a^2b^4} = \frac{8\cdot 9\cdot a^3a^4b^3b^2}{-1\cdot 36\cdot a\cdot a^2\cdot b^4} && \text{by rule 3}\\ &= \frac{72a^{3+4}b^{3+2}}{-36a^{1+2}b^4} = \frac{72a^7b^5}{-36a^3b^4} && \text{by rule 1}\\ &= \frac{-2a^7b^5}{a^3b^4} && \text{simplify numbers: } \frac{72}{-36} = \frac{-72}{36} = \frac{-2}{1}\\ &= \frac{-2a^{7-3}b^{5-4}}{1} = -2a^4b^1 = \boxed{-2a^4b} && \text{rule 2} \end{aligned}$$

2. Write each of the following expressions in terms of a fixed number or a single exponential expression.

a) $\dfrac{3^{2x+1}}{9^{x-1}}$

Solution: We will re-write the denominator in terms of base 3. After that, we can apply $\dfrac{a^n}{a^m} = a^{n-m}$.

$$\frac{3^{2x+1}}{9^{x-1}} = \frac{3^{2x+1}}{(3^2)^{x-1}} \overset{\text{Rule 3}}{=} \frac{3^{2x+1}}{3^{2(x-1)}} = \frac{3^{2x+1}}{3^{2x-2}} \overset{\text{Rule 2}}{=} 3^{2x+1-(2x-2)} = 3^{2x+1-2x+2} = 3^3 = \boxed{27}$$

b) $\dfrac{\left(8^{b-2}\right)\left(2^{b+1}\right)}{4^{2b-3}}$

Solution: We will re-write each exponential expressions in terms of base 2.

$$\begin{aligned}\frac{\left(8^{b-2}\right)\left(2^{b+1}\right)}{4^{2b-3}} &= \frac{8^{b-2}\cdot 2^{b+1}}{4^{2b-3}} = \frac{\left(2^3\right)^{b-2}\cdot 2^{b+1}}{\left(2^2\right)^{2b-3}} \overset{\text{Rule 3}}{=} \frac{2^{3(b-2)}\left(2^{b+1}\right)}{2^{2(2b-3)}} = \frac{2^{3b-6}\cdot 2^{b+1}}{2^{4b-6}} \overset{\text{Rule 1}}{=} \frac{2^{3b-6+(b+1)}}{2^{4b-6}}\\ &= \frac{2^{4b-5}}{2^{4b-6}} \overset{\text{Rule 2}}{=} 2^{(4b-5)-(4b-6)} = 2^{4b-5-4b+6} = 2^1 = \boxed{2}\end{aligned}$$

c) $5^{2x-1}\cdot 25^{3-x}$

$$5^{2x-1}\cdot 25^{3-x} = 5^{2x-1}\cdot\left(5^2\right)^{3-x} \overset{\text{Rule 3}}{=} 5^{2x-1}\cdot 5^{2(3-x)} = 5^{2x-1}\cdot 5^{6-2x} = \overset{\text{Rule 1}}{=} 5^{2x-1+6-2x} = 5^5 = \boxed{3125}$$

3. Let us denote 3^{100} by M. Express each of the following in terms of M.

a) 3^{101}

Solution: Using rule 1, we write $3^{101} = 3^{100+1} = 3^{100}\cdot 3^1 = M\cdot 3 = \boxed{3M}$

b) $3^{100} - 2\cdot 3^{101} + 3^{102}$

Solution: Using rule 1, we re-write 3^{101} and 3^{102}

$$\begin{aligned}3^{101} &= 3^{100+1} = 3^{100}\cdot 3^1 = M\cdot 3 = 3M\\ 3^{102} &= 3^{100+2} = 3^{100}\cdot 3^2 = M\cdot 9 = 9M\end{aligned}$$

Then our expression becomes

$$3^{100} - 2\cdot 3^{101} + 3^{102} = M - 2\cdot(3M) + 9M = M - 6M + 9M = -5M + 9M = \boxed{4M}$$

c) 3^{99}

Solution: Using rule 2, we write $3^{99} = 3^{100-1} = \dfrac{3^{100}}{3^1} = \boxed{\dfrac{M}{3}}$

d) 9^{100}

Solution: This time we will use rule 3 in a novel way: $(a^n)^m = (a^m)^n$

$$9^{100} = \left(3^2\right)^{100} = \left(3^{100}\right)^2 = M^2$$

We can also solve this problem using rule 4

$$9^{100} = (3\cdot 3)^{100} = 3^{100}\cdot 3^{100} = M\cdot M = \boxed{M^2}$$

4. Find the prime-factorization for each of the following numbers.

a) 10^{2018}

Solution: We will find the prime-factorization of the base and then apply rules of exponents.

$10^{2018} = (2 \cdot 5)^{2018} = \boxed{2^{2018} \cdot 5^{2018}}$

b) 18^{1000}

Solution: We will find the prime-factorization of the base and then apply rules of exponents.

$18^{1000} = (2 \cdot 9)^{1000} = \left(2 \cdot 3^2\right)^{1000} = 2^{1000} \cdot \left(3^2\right)^{1000} = 2^{1000} \cdot 3^{2 \cdot 1000} = \boxed{2^{1000} \cdot 3^{2000}}$

c) 360^{50}

Solution: We will find the prime-factorization of the base and then apply rules of exponents.

$360 = 36 \cdot 10 = (4 \cdot 9) \cdot (2 \cdot 5) = 2^3 \cdot 3^2 \cdot 5$. So the prime factorization of 360 is $2^3 \cdot 3^2 \cdot 5$.

$360^{50} = \left(2^3 \cdot 3^2 \cdot 5\right)^{50} = \left(2^3\right)^{50} \left(3^2\right)^{50} (5)^{50} = 2^{3 \cdot 50} \cdot 3^{2 \cdot 50} \cdot 5^{50} = \boxed{2^{150} \cdot 3^{100} \cdot 5^{50}}$

d) The product $10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$ comes up a lot in mathematics, so there is notation for it.

$10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 10!$ We pronounce $10!$ as ten factorial.

Now we just find the prime-factorization of each factor and collect the prime-factorization that way.

$$\begin{aligned} 10! &= 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 && \text{we drop 1 at the end} \\ &= (2 \cdot 5)\left(3^2\right)\left(2^3\right) \cdot 7 \cdot (2 \cdot 3) \cdot 5 \cdot 2^2 \cdot 3 \cdot 2 \end{aligned}$$

We collect the two-factors: one from 10, three from 8, one from 6, two from 4, and one from two. Similarly, the three-factors: two from 9, one from 6 and one from 3. Five factors come out only from 10 and 5, one from each, and the greatest prime factor is 7.

$$\begin{aligned} 10! &= (2 \cdot 5)\left(3^2\right)\left(2^3\right) \cdot 7 \cdot (2 \cdot 3) \cdot 5 \cdot 2^2 \cdot 3 \cdot 2 \\ &= 2^{1+3+1+2+1} \cdot 3^{2+1+1} \cdot 5^{1+1} \cdot 7 = \boxed{2^8 \cdot 3^4 \cdot 5^2 \cdot 7} \end{aligned}$$

5. Re-write each of the given numbers using scientific notation.

a) 3800 000 000

Solution: 3800 000 000 ends in eight zeroes. This can be translated as

$3800\,000\,000 = 38 \cdot 10^8$. We are not done yet: the first factor is too big, it must be between 1 and 10. So we re-write 38 as $3.8 \cdot 10$ and use the rules of exponents:

$3800\,000\,000 = 38 \cdot 10^8 = (3.8 \cdot 10) \cdot 10^8 = \boxed{3.8 \cdot 10^9}$

b) 6250 000 000 000

Solution: We count ten trailing zeroes, so

$6250\,000\,000\,000 = 625 \cdot 10^{10}$. We re-write 625 as $6.25 \cdot 10^2$. So

$6250\,000\,000\,000 = 625 \cdot 10^{10} = \left(6.25 \cdot 10^2\right) \cdot 10^{10} = \boxed{6.25 \cdot 10^{12}}$

6. Suppose that $A = 5 \cdot 10^{18}$, and $B = 8 \cdot 10^7$. Compute each of the following. Present your answer using scientific notation.

a) AB b) A^2 c) $2A$ d) $\dfrac{4A}{B}$

Solution: $AB = \left(5 \cdot 10^{18}\right)\left(8 \cdot 10^7\right) = (5 \cdot 8)\left(10^{18} \cdot 10^7\right) = 40 \cdot 10^{25} = \boxed{4 \cdot 10^{26}}$

b) A^2

Solution: $A^2 = \left(5 \cdot 10^{18}\right)^2 = 5^2 \cdot \left(10^{18}\right)^2 = 25 \cdot 10^{18 \cdot 2} = 25 \cdot 10^{36} = 2.5 \cdot 10 \cdot 10^{36} = \boxed{2.5 \cdot 10^{37}}$

c) $2A$

Solution: $2A = 2\left(5 \cdot 10^{18}\right) = (2 \cdot 5) \cdot 10^{18} = 10 \cdot 10^{18} = \boxed{1 \cdot 10^{19}}$

d) $\dfrac{4A}{B}$

Solution: $\dfrac{4A}{B} = \dfrac{4\left(5 \cdot 10^{18}\right)}{8 \cdot 10^7} = \dfrac{20 \cdot 10^{18}}{8 \cdot 10^7}$

Unfortunately, 8 is not a divisor of 20, but it is a divisor of 200. So, we borrow a ten from the ten-power.

$$\frac{20 \cdot 10^{18}}{8 \cdot 10^7} = \frac{200 \cdot 10^{17}}{8 \cdot 10^7} = 25 \cdot 10^{10} = \boxed{2.5 \cdot 10^{11}}$$

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