Grouping is a factoring technique that can be used in several situations. Grouping consists of strategically pairing (or grouping) terms, and then factoring out the greatest common factor or GCF three times. We can also say that factoring by grouping is a reversal of FOIL (First, Outer, Inner, Last). We should consider factoring by groupimg when we have four terms and perhaps also several variables or higher degrees.

Example 1. Completely factor the expression $15ax + 6ay - 10bx - 4by$.

Solution: The first step is grouping the four terms into two pairs. The goal is to pair terms that have similar terms or coefficients with common divisors. The goal is to have as much of a GCF in a pair as possible.

In this case, we have two options that would both work. We could pair $15ax$ with $6ay$ because they share the common factor of $3a$. We could also pair $15ax$ with $-10bx$ because then $5x$ is a shared factor. The only pairing that would not work is to pair $15ax$ with $4by$ as these two terms have no common factor besides 1.

So, we first pair the first two terms and the second two tems. As we stated before, grouping is to factor out the common factor three times. First, we factor out the greatest common factor from $15ax + 6ay$.

$$3a\left(5x+2y\right) - 10bx - 4by$$

This method can only work if, when factoring out the GCF from the second pair, what is left in the parentheses is identical to $5x + 2y$. The greatest common factor of $-10bx - 4by$ is $2b$. This means that we have two choices: either factor out $2b$ or $-2b$. We have to select the sign that guarantees that we have $5x+2y$ left in the parentheses. Therefore, we factor out $-2b$ from $-10bx - 4by$.

$$3a\left(5x+2y\right) - 2b\left(5x+2y\right)$$

At this point, $5x + 2y$ is the common factor of the two terms and so we can factor that out.

$$3a\underbrace{\left(5x+2y\right)}_{\text{GCF}} - 2b\underbrace{\left(5x+2y\right)}_{\text{GCF}} = \left(5x+2y\right)\left(3a-2b\right)$$

So the factored form is $\boxed{\left(5x+2y\right)\left(3a-2b\right)}$. When we check, we can see why this method is a reversal of FOIL:

$$\left(5x+2y\right)\left(3a-2b\right) = 15ax - 10bx + 6ay - 4by$$

The next example illustrates a commonly occuring situation, when one or more of the GCF is 1. Still, grouping will work just fine.

Example 2. Completely factor the expression $6mx - 3x - 2m + 1$.

Solution: The first step is grouping the four terms into two pairs. The goal is to pair terms that have similar terms or coefficients with common divisors. The goal is to have as much of a GCF in a pair as possible. The terms here seem to share not much in common, but the first and third term share $2m$. So, we first rearrange the terms:

$$\begin{aligned} 6mx - 3x - 2m + 1 &= 6mx - 2m - 3x + 1 \quad \text{the GCF in the first two terms is } 2m \\ &= 2m\left(3x-1\right) - 3x + 1 \end{aligned}$$

The second pair shares no factor besides 1, but we already see the similarity between the expression left after factoring out the GCF from the first pair and the second pair. The only quation is, should we factor out 1 or -1?

The goal is to have the same expression left in the parentheses. Therefore, we should factor out -1.

$$2m(3x-1)-3x+1=2m(3x-1)-1(3x-1)$$

Now $3x-1$ is the common factor.

$$2m(3x-1)-1(3x-1)=(2m-1)(3x-1)$$

So the factored form is $\boxed{(2m-1)(3x-1)}$. When we check, we can see why this method is a reversal of FOIL:

$$(2m-1)(3x-1)=6mx-2m-3x+1$$

Sometimes we see only one variable, but with higher degrees. In this case, having four terms is still an important indication that grouping would work.

Example 3. Completely factor the expression $2\left(x^2-4\right)(5x+3)=10x^3+6x^2-40x-24$.

Solution: Like with all factoring, we start with the GCF. In this case, all terms are divisible by 2, so we will factor it out. Then we group the terms by degrees.

$$\begin{aligned} 10x^3+6x^2-40x-24 &= 2\left(5x^3+3x^2-20x-12\right) && \text{factor out } x^2 \text{ from first pair}\\ &= 2\left[x^2(5x+3)-20x-12\right] && \text{factor out } -4 \text{ from second pair}\\ &= 2\left[x^2(5x+3)-4(5x+3)\right] && \text{factor out } (5x+3)\\ &= 2\left(x^2-4\right)(5x+3) \end{aligned}$$

Although we are done with factoring by grouping, the expression $2\left(x^2-4\right)(5x+3)$ is not *completely* factored, because x^2-4 can be factored by the difference of squares theorem into $(x+2)(x-2)$. So the factored form is $\boxed{2(x+2)(x-2)(5x+3)}$. We can check by multiplying back:

$$2(x+2)(x-2)(5x+3)=2\left(x^2-4\right)(5x+3)=2\left(5x^3+3x^2-20x-12\right)=10x^3+6x^2-40x-24$$

and so our solution is correct.

Another important application of grouping is factoring a general quadratic trinomial, ax^2+bx+c.

Example 4. Completely factor $14x^2-8x+21x-12$.

Solution: We group $14x^2$ with $21x$ and $-8x$ with -12.

$$\begin{aligned} 14x^2-8x+21x-1 &= 14x^2+21x-8x-12 && \text{factor out GCF from first pair}\\ &= 7x(2x+3)-8x-12 && \text{factor out GCF from second pair}\\ &= 7x(2x+3)-4(2x+3) && \text{factor ou the GCF } 2x+3\\ &= (2x+3)(7x-4) \end{aligned}$$

So the factored form is $\boxed{(2x+3)(7x-4)}$. We can check by multiplying back:

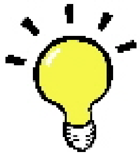
$$(2x+3)(7x-4)=14x^2-8x+21x-12$$

Note that general trinomials usually do not occur with two like terms such as $-8x + 21x$. So, how could we factor $14x^2 + 13x - 12$ by grouping? The trick is to strategically 'take apart' the linear term $13x$ to two like terms $-8x + 21x$. But how do we know how to take apart the linear term? There is a systematic way to do that, and it will be discussed later. This factoring technique is called the AC-method.

Practice Problems

Completely factor each of the following.

1. $5x - 6a + 2ax - 15$
2. $2a - 4x^2 + 2ax^2 - 4$
3. $5ax^3 - 5bx^3 + 5ax^2y - 5bx^2y$
4. $3a - 3ax - 3ay + 3axy$
5. $18m - 90n - 2mx^2 + 10nx^2$
6. $p^2x^2 - p^2y^2 - q^2x^2 + q^2y^2$
7. $2a^2by - 6a^2bt - 3a^2btx + a^2bxy$
8. $2a^2b - 50b - 25bm^3 + a^2bm^3$
9. $6x^5 - 15x^4 + 2x - 5$
10. $-24ax^9 + 8ax^7 + 6ax^3 - 2ax$
11. $x^3 + 2x^2 - 4x - 8$
12. $6x^2 + 16x - 3x - 8$
13. $2x^2 + 5x - 6x - 15$
14. $x^2 - 2x - 4x + 8$
15. $6x^2 + 4x - 3x - 2$
16. $10x^2 - 25x - 4x + 10$



Answers

Practice Problems

1. $(x-3)(2a+5)$ 2. $2(a-2)\left(x^2+1\right)$ 3. $5x^2(a-b)(x+y)$ 4. $3a(x-1)(y-1)$

5. $2(x+3)(x-3)(5n-m)$ 6. $(x+y)(x-y)(p+q)(p-q)$ 7. $a^2b(x+2)(y-3t)$

8. $b(a+5)(a-5)\left(m^3+2\right)$ 9. $\left(3x^4+1\right)(2x-5)$ 10. $-2ax\left(3x^2-1\right)\left(2x^3+1\right)\left(2x^3-1\right)$

11. $(x-2)(x+2)^2$ 12. $(3x+8)(2x-1)$ 13. $(2x+5)(x-3)$ 14. $(x-2)(x-4)$

15. $(3x+2)(2x-1)$ 16. $(5x-2)(2x-5)$

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