

Example 1. In a local diner, we are about to order a three-entree meal. We can choose from 3 different soups, 2 main dishes, and 4 desserts. How many different meal can we order?

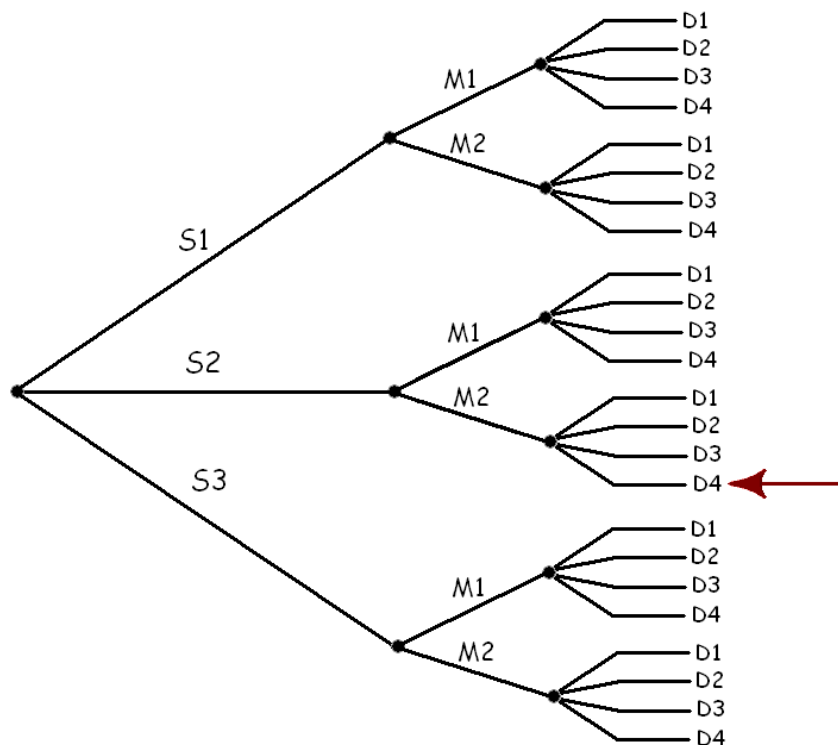
Solution: Suppose we can select soup S1, S2, and S3, main dish M1, or M2, and desserts D1, D2, D3, and D4. If we decide to go with soup 1, we still have 2 options for the main dish and 4 for the dessert. We can select M1 with D1, D2, D3, or D4 - that's 4 options, or, we could select M2 with D1, D2, D3, or D4 - so these are 8 different options. If we changed our mind and went with soup S2, we had the same 8 options. And, the same 8 options with soup S3. Therefore, we can assemble 24 different meals. We can list the possible meals:

	S	M	D
1	1	1	1
2	1	1	2
3	1	1	3
4	1	1	4
5	1	2	1
6	1	2	2
7	1	2	3
8	1	2	4

	S	M	D
9	2	1	1
10	2	1	2
11	2	1	3
12	2	1	4
13	2	2	1
14	2	2	2
15	2	2	3
16	2	2	4

	S	M	D
17	3	1	1
18	3	1	2
19	3	1	3
20	3	1	4
21	3	2	1
22	3	2	2
23	3	2	3
24	3	2	4

We count our list and find that the answer is 24. There are 24 different meals can be assembled. To represent these outcomes, we could also use a tree.



The outcomes can be read as the end points of the tree. They can be interpreted by reading the branch from the root. For example, the ending marked with the arrow corresponds to the outcome soup 2, main dish 2, and dessert 4.

Theorem: The **Fundamental Counting Principle** states that if one thing can be done in n many different ways, and another thing, independently, can be done in k many different ways, then both can be done in nk many different ways.

This principle only works if the first choice does not affect the second choice, if they are truly independent. For example, if one of the main dishes is fried rice, we may not want to select rice pudding for dessert. In this case, the fundamental counting principle would not apply here. It only works if we have the same number of options for the main dish, no matter what soup we selected, and we have the same number of options for dessert, no matter what soup and main dish we selected before.

If we toss a coin, it has an even chance to fall into one of its two different sides.

These two outcomes are called heads and tails.



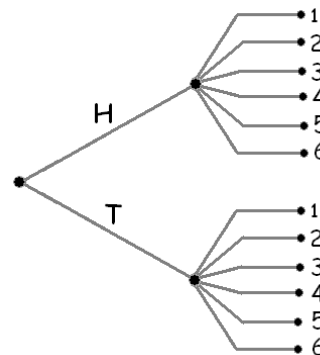
When we roll a die (or two dice), each has an even chance to fall to show 1, 2, 3, 4, 5, or 6 on its top face.

Example 2. We toss a coin (the possible outcomes are head or tail) and then throw a die (with possible outcomes 1, 2, 3, 4, 5, and 6) What is the number of possible outcomes?

Solution: We have 2 options for the coin toss, and 6 for the die. Because the two are independent, this means 12 outcomes by the fundamental counting principle.

If the outcome of the coin toss is heads (H), there are 6 possible outcomes of the roll of the die. The same is true with tail (T) result for the coin toss. We can list the outcomes, or draw a tree.

H1	T1
H2	T2
H3	T3
H4	T4
H5	T5
H6	T6



Example 3. In how many different orders can we list the letters A, B, and C?

Solution: In small cases like 3 letters, we can solve the problem by systematic listing. If A was replaced by 1, B by 2 and C by 3, we could list them in an increasing order. That is the same as listing in an alphabetical order.

begins with A	begins with B	begins with C
ABC	BAC	CAB
ACB	BCA	CBA

Notice that our list is not written in a single line: ABC, ACB, BAC, BCA, CAB, CBA. The visual organization is a tool we can use to our advantage. In this case, it follows the logic and method of organizing,

The first column counts all orders that start with A. The second column lists all orders that start with B, and orders starting with C in the third column. We count that, there are 6 different ways we can list 3 objects. This visual organization might seem unnecessary here, but it will be very helpful with larger numbers for n .

Example 4. In how many different orders can we list the letters A, B, C, and D?

Solution: We can also list the possibilities with four objects. Again, we will order our list alphabetically, and will be slick about visual organization.

begins with A	begins with B	begins with C	begins with D
ABCD	BACD	CABD	D ABC
ABDC	BADC	CADB	D ACB
ACBD	BCAD	CBAD	D BAC
ACDB	BCDA	CBDA	D BCA
ADBC	BDAC	CDAB	D CAB
ADCB	BDCA	CDBA	D CBA

For the first column, we will list all orders that start with A. The second column will list all orders starting with a B, the third column for C, and the fourth for D. Therefore, we will have four columns. Consider now the fourth column, all orders that begin with D. If we fix the first letter to be D, then the different possible orders are just rearrangements of ABC. Similarly, the first column reflects rearrangements of BCD, given that A is fixed as the first letter. The same can be stated about the second column (rearrangements ACD) and the third column (with ABD). Therefore, each column is 6 long. So we have that there are $6 \cdot 4 = 24$ different ways we can arrange 4 objects.

We can continue using this technique. For 5 objects, we organize them into columns, where each column contains all arrangements that begin with A, B, C, D, and E. Therefore, we will have 5 columns. Each column will reflect the number of rearrangements of the remaining 4 objects - that we just found to be 24. Therefore, we will have 5 columns, each 24 long. So, there are $5 \cdot 24 = 120$ possible orders. Similarly, for 6 objects, we will have 6 columns, each 120 long, and so on.

The same questions can also be answered via the fundamental counting principle. Instead of listing all possible outcomes, we 'pretend' to create one, while following how many choices we had.

Example 5. In how many different orders can we list the letters A, B, C, D, and E?

Solution: Suppose that we are selecting one such possible arrangement, letter by letter. First, we select the first letter. For that, we have 5 options, we can pick any of A, B, C, D, and E. For the second letter, we will have only 4 choices left, because whatever we selected first is no longer an option.. For the third letter, we have only 3 options left. For the fourth letter, we have only 2 options left. Finally, we have only 1 option left for the last letter. By the fundamental counting principle, this means $5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$ possible outcomes.

What is interesting in this problem, is that the second choice is not exactly independent of the first choice, like it was in our meal selection. If we selected C for the first letter, we cannot select C again for the second letter. However, no matter what the choice for the first letter was, the *number of options* is the same for the second letter.

Theorem: There are $n \cdot (n - 1) \cdot (n - 2) \dots 3 \cdot 2 \cdot 1$ possible ways to order n objects. The different orders of n objects is called a **permutation** on those objects.

This repeated multiplication of smaller and smaller integers often comes up in combinatorics, so there is a notation developed for it.

Definition: If n is a positive integer, $n \geq 2$, then $n!$ (pronounced n **factorial**) is the product $n(n-1)(n-2)\dots\cdot 3\cdot 2\cdot 1$. We also define $0!$ and $1!$ both to have value 1.

The number of permutations on n objects is $n!$

In addition to permutations, the fundamental counting principle can be used for other type of counting. For example, if we are counting possible outcome in which repetition is allowed.

Example 6. How many four-digit numbers can be formed using only the digits 2, 3, 5, 7, and 8?

Solution: Considering that 2822 is a four-digit number, this time we also need to count numbers in which digits are repeating. This makes solving this problem easier. For the first digit, we have 5 options: the number 2, 3, 5, 7, and 8. For the second digit, we have the same 5 digits available. For the four digits, we have each time 5 options. Therefore, the number of possible numbers is

$$N = 5 \cdot 5 \cdot 5 \cdot 5 = 625$$

When we are ordering objects, each can only be selected once. When we are writing numbers, digits can be repeated. This is an important distinction. If every object can be used only once, we say we count outcomes **without replacement**. If an object can be selected several times in one outcome, we count with **replacement**.

Example 7. Using only non-zero digits, how many different three-digit numbers are there if

- a) repetition of digits is allowed b) repetition of digits is not allowed.

Solution: a) If repetition of digits is allowed, then we have all 9 digits available as selection for each digit.

$$N = 9 \cdot 9 \cdot 9 = 729$$

- b) If repetition of digits is not allowed, then our options decrease with each choice we make. For the first digit, we have 9 choices. For the second one, we have 8 options, and 7 for the third one.

$$N = 9 \cdot 8 \cdot 7 = 504$$

Example 8. Five married couple will be seated at long table. How many different arrangements are possible

- a) with no restriction b) if married couples are to be seated next to each other?

Solution: a) This is a classic case of permutations on 10 objects. The answer is $10! = 3628800$.

- b) Imagine that we are deciding on the seating. Let us first decide, which couple takes seats 1 and 2, 2 and 3, and so on. For the first pair of seats, we can select one out of 5 couples. For the second pair of seats, we can select one out of 4 couples, and so on. So, just focusing at the couples, we have $5!$ choices. In each arrangement, within a couple, AB and BA are the possible seatings. Therefore, the answer is

$$N = \underset{\text{first couple}}{5 \cdot 2} \cdot \underset{\text{second couple}}{4 \cdot 2} \cdot \underset{\text{third couple}}{3 \cdot 2} \cdot \underset{\text{fourth couple}}{2 \cdot 2} \cdot \underset{\text{fifth couple}}{1 \cdot 2} = 5! \cdot 2^5 = 32 \cdot 5! = 3840$$

Example 9. Twenty students will compete in the yearly championship. How many different outcome is possible for the first, second, and third place?

Solution: For the first place, there are 20 student candidates. Once that is selected, the second place can be selected from the remaining 19 students. For the third place, only 18 left. Therefore,

$$N = 20 \cdot 19 \cdot 18 = 6840$$

Example 10. A licence plate is to contain two letters (out of 26) and four numbers. How many different plates are possible?

Solution: By the fundamental counting principle, $N = 26 \cdot 26 \cdot 10 \cdot 10 \cdot 10 \cdot 10 = 6760\,000$



Practice Problems

1. We roll two dice. How many outcomes are possible?
2. We toss a coin five times in a row. How many outcomes are possible?
3. Let us agree that in this class, no positive interger starts with the digit zero. Then how many six-digit numbers are there
 - a) all together
 - b) with all digits are being different
4. We have 10 different letters addressed to 10 different people. We also have 10 addressed envelopes. In how many ways can we put the letters in the envelope? (We put one letter in each envelope.)
5. We have a class of 12 people, 6 males and 6 females.
 - a) In how many ways can they be seated at a long table?
 - b) In how many ways can they be seated at a long table so that there is only one female sitting next to a male
6.
 - a) In how many different ways can we list t all he letters in the word APPLE? List them to count.
 - B) Explain why the fundamental counting principle does not work here.
7. Teams A and B are the finalists in a competition. They will play games until one team wins three games. That team wins the championship.
 - a) How many games could be played in the final?
 - b) List all outcomes that result in team A winning the championship.



Answers

1. 36 2. $2^5 = 32$ 3. a) $9 \cdot 10^5 = 900\,000$ b) $9 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 = 136\,080$ 4. $10! = 3\,628\,800$

5. a) $12! = 479\,001\,600$ b) $2(6!)(6!) = 1036\,800$

6. a) 60

1	AELPP	EALPP	LAEPP	PAELP	PLAEP
2	AEPLP	EAPLP	LAPEP	PAEPL	PLAPE
3	AEPPL	EAPPL	LAPPE	PALEP	PLEAP
4	ALEPP	ELAPP	LEAPP	PALPE	PLEPA
5	ALPEP	ELPAP	LEPAP	PAPEL	PLPAE
6	ALPPE	ELPPA	LEPPA	PAPLE	PLPEA
7	APELP	EPALP	LPAEP	PEALP	PPAEL
8	APEPL	EPAPL	LPAPE	PEAPL	PPALE
9	APLEP	EPLAP	LPEAP	PELAP	PPEAL
10	APLPE	EPLPA	LPEPA	PELPA	PPELA
11	APPEL	EPPAL	LPPAE	PEPAL	PPLAE
12	APPLE	EPPLA	PLPEA	PEPLA	PPLEA

b) We have 1 of A, E, and L, but 2 of P. Because of this, the second choice is not independent from the first one. If we select P for first digit, we have 4 options for the second letter. But if we select A for the first letter, we have only 3 choices for the second letter. Therefore, the fundamental principle will not work here.

7. a) 3 or 4 or 5

b) B wins zero games: AAA B wins one game: BAAA, ABAA, AABA B wins two games: BBAAA, BABAA, BAABA, ABBAA, ABABA