

Completing the square is a powerful and elegant factoring technique. This technique is fundamentally different from other factoring techniques such as grouping or trial and error. While those other techniques enable us to factor over the integers, completing the square enables us to factor over the real numbers. If we wanted to factor expressions over the real numbers and the numbers turn out to be irrational, then grouping and trial and error will not work. Completing the square always does.

Completing the square is not just a factoring technique. We will soon see that it is also a way to understand quadratic expressions.

## Part 1 - When the Leading Coefficient is 1

**Definition:** A **complete square** is an algebraic expression in which a sum (or difference) is being squared. For example,  $(x - 2)^2$ ,  $(2a + 3)^2$ , and  $(a + b - c)^2$  are complete squares.

**Example 1.** Expand each of the given complete squares.

a)  $(x - 2)^2$       b)  $(2a + 3)^2$

**Solution:** a)  $(x - 2)^2 = (x - 2)(x - 2) = x^2 - 2x - 2x + 4 = \boxed{x^2 - 4x + 4}$

b)  $(2a + 3)^2 = (2a + 3)(2a + 3) = 4a^2 + 6a + 6a + 9 = \boxed{4a^2 + 12a + 9}$

**Example 2.** Factor  $-4x + x^2 - 21$  by completing the square.

**Solution:** Step 1. We re-arrange the terms by decreasing order of degree.

$$-4x + x^2 - 21 = x^2 - 4x - 21$$

Step 2A. We obtain the "magic number", that is half of the linear coefficient. The linear coefficient is the number multiplying  $x$ , **sign included**. In our example, the magic number is  $\frac{-4}{2} = -2$ . We do not write this line in the main computation.

Step 2B. We place an  $x$  in front of the magic number, and square the expression we obtained. Work out this computation on the margin, not in the main computation.

$$(x - 2)^2 = (x - 2)(x - 2) = x^2 - 2x - 2x + 4 = x^2 - 4x + 4$$

Step 2C. We write the "helper line"  $(x - 2)^2 = x^2 - 4x + 4$  in the upper right hand side of the paper. We will use it twice. Our computation so far looks like this:

$$-4x + x^2 - 21 = x^2 - 4x - 21 \qquad (x - 2)^2 = x^2 - 4x \quad \boxed{+4}$$

Step 3. The smuggling step. What we have achieved in Step 2, is to have found the only perfect square that begins with the same two terms,  $x^2 - 4x$  as our expression to be factored. We can see that the last term,  $+4$  is missing. We complete the square as follows.

Step 3A. Write down our expression with one modification: we leave a gap between the second and third terms.

$$x^2 - 4x \quad - 21$$

Step 3B. We add zero to the expression by adding and then immediately subtracting 4 into the gap.

$$x^2 - 4x + 4 - 4 - 21$$

Step 4. We have obtained five terms. We re-write the first three terms as a perfect square (the second time we used the helper line) and combine the last two terms.

$$x^2 - 4x - 21 = \underbrace{x^2 - 4x + 4}_{(x-2)^2} - 4 - 21 = (x-2)^2 - 25$$

Step 5. We re-write the last number as a square.

$$(x-2)^2 - 25 = (x-2)^2 - 5^2$$

Step 6. If applies, we factor via the difference of squares theorem.

$$(x-2)^2 - 5^2 = (x-2+5)(x-2-5)$$

Step 7. (Cleanup) We simplify the factors by combining like terms.

$$(x-2+5)(x-2-5) = \boxed{(x+3)(x-7)}$$

Step 8. We check our result by multiplication.

$$(x+3)(x-7) = x^2 - 7x + 3x + 21 = x^2 - 4x + 21$$

Thus our result,  $(x+3)(x-7)$  is correct.

The entire computation should look like this:

$$\begin{aligned} -4x + x^2 - 21 &= \\ &= x^2 - 4x - 21 & (x-2)^2 &= x^2 - 4x + \boxed{4} \\ &= \underbrace{x^2 - 4x + 4}_{(x-2)^2} - 4 - 21 \\ &= (x-2)^2 - 25 \\ &= (x-2)^2 - 5^2 \\ &= (x-2+5)(x-2-5) \\ &= \boxed{(x+3)(x-7)} \end{aligned}$$

$$\text{We check: } (x+3)(x-7) = x^2 - 7x + 3x - 21 = x^2 - 4x - 21$$

**Example 3.** Factor  $32 - 18x + x^2$  by completing the square.

**Solution:** Step 1. We re-arrange the terms by decreasing order of degree.

$$32 - 18x + x^2 = x^2 - 18x + 32$$

Step 2A. We obtain the "magic number", that is half of the linear coefficient. (The linear coefficient is the number multiplying  $x$ , sign included.) In our example, the magic number is  $\frac{-18}{2} = -9$ .

Step 2B. We place an  $x$  in front of the magic number, and square the expression we obtained. Do not write this computation in the main computation.

$$(x - 9)^2 = (x - 9)(x - 9) = x^2 - 9x - 9x + 81 = x^2 - 18x + 81$$

Step 2C. We write the "helper line"  $(x - 9)^2 = x^2 - 18x + 81$  in the upper right hand side of the paper.

$$x^2 - 18x + 32 \qquad (x - 9)^2 = x^2 - 18x + \boxed{81}$$

Step 3. (The smuggling step.) What we have achieved in Step 3, is to have found the only perfect square that begins with the same two terms,  $x^2 - 18x$  as our expression to be factored. We can see that the last term,  $+81$  is missing. We complete the square by writing down our expression with a gap between the second and third terms, and then adding zero to the expression by adding and then immediately subtracting 81 in the gap.

$$x^2 - 18x + 32 = x^2 - 18x + 81 - 81 + 32$$

Step 4. We obtained five terms. We re-write the first three terms as a perfect square and combine the last two terms.

$$x^2 - 18x + 32 = \underbrace{x^2 - 18x + 81}_{(x-9)^2} - 81 + 32 = (x - 9)^2 - 49$$

Step 5. We re-write the last number as a square.

$$(x - 9)^2 - 49 = (x - 9)^2 - 7^2$$

Step 6. If applies, we factor via the difference of squares theorem.

$$(x - 9)^2 - 7^2 = (x - 9 + 7)(x - 9 - 7)$$

Step 7. (Cleanup) We simplify the factors by combining like terms.

$$(x - 9 + 7)(x - 9 - 7) = (x - 2)(x - 16)$$

Step 8. We check back by multiplication.

$$(x - 2)(x - 16) = x^2 - 2x - 16x + 32 = x^2 - 18x + 32$$

The entire computation should look like this:

$$\begin{aligned}
 32 - 18x + x^2 &= x^2 - 18x + 32 & (x - 9)^2 &= x^2 - 18x \boxed{+ 81} \\
 &= \underbrace{x^2 - 18x + 81}_{(x-9)^2} - 81 + 32 \\
 &= (x - 9)^2 - 49 \\
 &= (x - 9)^2 - 7^2 \\
 &= (x - 9 + 7)(x - 9 - 7) \\
 &= \boxed{(x - 2)(x - 16)}
 \end{aligned}$$

We check:  $(x - 2)(x - 16) = x^2 - 2x - 16x + 32 = x^2 - 18x + 32$

Thus our result,  $(x - 2)(x - 16)$  is correct.

**Example 4.** Factor  $28x + x^2 - 1173$  by completing the square.

**Solution:** We first rearrange the terms by degrees.

$$28x + x^2 - 1173 = x^2 + 28x - 1173 \quad \text{half of the linear coefficient is } \frac{28}{2} = 14$$

We work out  $(x + 14)^2 = x^2 + 28x + 196$  on the margin. So we know to smuggle in 196.

$$\begin{aligned}
 &= x^2 + 28x - 1173 & (x + 14)^2 &= x^2 + 28x \boxed{+ 196} \\
 &= \underbrace{x^2 + 28x + 196}_{(x+14)^2} - 196 - 1173 & \text{realize comple square, combine like terms} \\
 &= (x + 14)^2 - 1369 & \text{from calculator, } \sqrt{1369} &= 37 \\
 &= (x + 14)^2 - 37^2 & \text{difference of squares theorem} \\
 &= (x + 14 + 37)(x + 14 - 37) & \text{combine like terms} \\
 &= \boxed{(x + 51)(x - 23)}
 \end{aligned}$$

We check by multiplication:  $(x + 51)(x - 23) = x^2 - 23x + 51x - 1173 = x^2 + 28x - 1173$

Thus our result,  $(x + 51)(x - 23)$  is correct.

**Example 5.** Factor  $17 - 2a + a^2$  by completing the square.

**Solution:** We first rearrange the terms by degree.

$$\begin{aligned}
 17 - 2a + a^2 &= 17 - 2a + a^2 & \text{rearrange terms} \\
 &= a^2 - 2a + 17 & \text{the "magic number" is } \frac{-2}{2} = -1
 \end{aligned}$$

We work out  $(a - 1)^2 = a^2 - 2a + 1$  on the margin.

$$\begin{aligned}
 &= a^2 - 2a + 17 & (a - 1)^2 &= a^2 - 2a \boxed{+ 1}, \text{ so we smuggle in 1} \\
 &= \underbrace{a^2 - 2a + 1}_{(a-1)^2} - 1 + 17 & \text{realize comple square, combine like terms} \\
 &= (a - 1)^2 + 16
 \end{aligned}$$

We can not apply the difference of squares theorem, since 16 is added, not subtracted. **The sum of squares can not be factored**, and so the expression  $a^2 - 2a + 17$  can not be factored.



## Practice Problems

Factor each of the following by completing the square.

1.  $x^2 - 10x + 21$

6.  $b^2 - 10b + 26$

11.  $m^2 - 42m + 432$

16.  $q^2 - 2q - 48$

2.  $x^2 - 6x + 8$

7.  $3 + x^2 - 4x$

12.  $x^2 - 50x + 525$

17.  $x^2 - 18x + 81$

3.  $22y + y^2 + 105$

8.  $d^2 + 2d + 2$

13.  $10y + y^2 - 375$

18.  $t^2 - 36t - 4437$

4.  $b^2 - 4b - 45$

9.  $6x + x^2 - 432$

14.  $x^2 - 40x + 336$

19.  $x^2 - 46x + 360$

5.  $14a + a^2 - 51$

10.  $x^2 - 14x + 58$

15.  $x^2 - 6x + 25$

20.  $14q + q^2 - 2352$

## Part 2 - When the Leading Coefficient is Not 1 but Can be Factored Out

Factoring by completing the square is an extremely powerful factoring technique. We will see later that this is the only method that does not break down once numbers stop being "nice". In this section, we will see expressions in which the leading coefficient is not 1, but it can be easily factored out.

**Example 6.** Factor  $18x - 3x^2 + 165$  by completing the square.

**Solution:** Step 1. We re-arrange the terms by decreasing order of degree.

$$18x - 3x^2 + 165 = -3x^2 + 18x + 165$$

Step 2. We factor out the greatest common factor.

$$-3x^2 + 18x + 165 = -3(x^2 - 6x - 55)$$

Step 3. We factor the expression within the parentheses by completing the square.

$$\begin{aligned} -3(x^2 - 6x - 55) &= (x - 3)^2 = x^2 - 6x + 9 \\ &= -3\left(\underbrace{x^2 - 6x + 9}_{(x-3)^2} - 9 - 55\right) \\ &= -3\left((x - 3)^2 - 64\right) \\ &= -3\left((x - 3)^2 - 8^2\right) \\ &= -3(x - 3 + 8)(x - 3 - 8) = \boxed{-3(x + 5)(x - 11)} \end{aligned}$$

Step 4. We check our result by multiplication.

$$-3(x + 5)(x - 11) = -3(x^2 - 11x + 5x - 55) = -3(x^2 - 6x - 55) = -3x^2 + 18x + 165$$

Thus our result,  $-3(x + 5)(x - 11)$  is correct.

**Example 7.** Factor  $267x^2 - 48x^3 + 3x^4$  by completing the square.

**Solution:** We first rearrange the terms by degree.

$$\begin{aligned}
 267x^2 - 48x^3 + 3x^4 &= \\
 &= 3x^4 - 48x^3 + 267x^2 && \text{factor out } 3x^2 \\
 &= 3x^2 (x^2 - 16x + 89) && (x - 8)^2 = x^2 - 16x \boxed{+ 64} \\
 &= 3x^2 \left( \underbrace{x^2 - 16x + 64}_{(x-8)^2} - 64 + 89 \right) && \text{realize complete square, combine like terms} \\
 &= 3x^2 \left( (x - 8)^2 + 25 \right)
 \end{aligned}$$

We can not apply the difference of squares theorem, since 25 is added, not subtracted. The sum of squares cannot be factored. Therefore, the completely factored form of the expression is  $\boxed{3x^2 (x^2 - 16x + 89)}$

**Example 8.** Factor  $9 - y^2 - 8y$  by completing the square.

**Solution:** We first rearrange the terms by degree and then factor out the leading coefficient.

$$\begin{aligned}
 9 - y^2 - 8y &= -y^2 - 8y + 9 \\
 &= -1 (y^2 + 8y - 9) && (y + 4)^2 = y^2 + 8y \boxed{+ 16} \\
 &= - \left( \underbrace{y^2 + 8y + 16}_{(y+4)^2} - 16 - 9 \right) && \text{realize complete square, combine like terms} \\
 &= - \left( (y + 4)^2 - 25 \right) && \text{re-write 25 as a square} \\
 &= - \left( (y + 4)^2 - 5^2 \right) && \text{factor via the difference of squares theorem} \\
 &= - (y + 4 + 5) (y + 4 - 5) && \text{combine like terms, drop extra parentheses} \\
 &= \boxed{- (y + 9) (y - 1)}
 \end{aligned}$$

We check:  $-(y + 9)(y - 1) = -(y^2 - y + 9y - 9) = -(y^2 + 8y - 9) = -y^2 - 8y + 9$

Thus our result,  $-(y + 9)(y - 1)$  is correct.



## Practice Problems

Completely factor each of the following by completing the square.

1.  $4x + 2x^2 - 30$

5.  $18c - 24c^2 + 6c^3$

9.  $10abc - 600ac + 5ab^2c$

2.  $70a^2 - 255a + 5a^3$

6.  $-2d - 2d^2 - d^3$

10.  $70y^3 + 24y^4 + 2y^5$

3.  $78b^2 - 30b^3 + 3b^4$

7.  $432 - x^2 - 6x$

11.  $18x^2y^2 - 216x^2y + 3x^2y^3$

4.  $32x + 2x^2 - 594$

8.  $x^2 - 14x + 58$

12.  $1000x - 50x^2 - 5x^3$

## Part 3 - With Fractions

It is strongly recommended to use fractions and not decimals. The steps are identical as before, the only complications are that we need to perform the same computations with fractions.

**Example 9.** Factor  $54x - 6x^2 + 60$  by completing the square.

$$\begin{aligned} 54x - 6x^2 + 60 &= && \text{rearrange terms} \\ -6x^2 + 54x + 60 &= && \text{factor out } -6 \\ -6(x^2 - 9x - 10) &= && \text{the "magic number" is } \frac{-9}{2} = -\frac{9}{2} \end{aligned}$$

We work out  $\left(x - \frac{9}{2}\right)^2$  on the margin:

$$\left(x - \frac{9}{2}\right)^2 = \left(x - \frac{9}{2}\right)\left(x - \frac{9}{2}\right) = x^2 - \frac{9}{2}x - \frac{9}{2}x + \frac{81}{4} = x^2 - \frac{18}{2}x + \frac{81}{4} = x^2 - 9x + \frac{81}{4}$$

So we know to smuggle in the missing third term,  $\frac{81}{4}$ .

$$\text{our helper line: } \left(x - \frac{9}{2}\right)^2 = x^2 - 9x + \frac{81}{4}$$

$$-6(x^2 - 9x - 10) =$$

$$= -6\left(\underbrace{x^2 - 9x + \frac{81}{4}}_{\text{realize complete square, re-write 10 as } \frac{10}{1} = \frac{40}{4}} - \frac{81}{4} - \frac{10 \cdot 4}{1 \cdot 4}\right)$$

$$= -6\left(\left(x - \frac{9}{2}\right)^2 - \frac{81}{4} - \frac{40}{4}\right) \quad \text{combine like terms re-write } \frac{121}{4} \text{ as } \left(\frac{11}{2}\right)^2$$

$$= -6\left(\left(x - \frac{9}{2}\right)^2 - \frac{121}{4}\right) \quad \text{factor via the difference of squares theorem}$$

$$= -6\left(\left(x - \frac{9}{2}\right)^2 - \left(\frac{11}{2}\right)^2\right) \quad \text{combine like terms simplify fractions}$$

$$= -6\left(x - \frac{9}{2} + \frac{11}{2}\right)\left(x - \frac{9}{2} - \frac{11}{2}\right)$$

$$= -6\left(x + \frac{2}{2}\right)\left(x - \frac{20}{2}\right)$$

$$= \boxed{-6(x + 1)(x - 10)}$$

We check by multiplication:

$$-6(x + 1)(x - 10) = -6(x^2 - 10x + x - 10) = -6(x^2 - 9x - 10) = -6x^2 + 54x + 60$$

Thus our result,  $-6(x + 1)(x - 10)$  is correct.

**Example 10.** Factor  $5p + p^2 + 6$  by completing the square.

$$5p + p^2 + 6 = \text{rearrange terms}$$

$$p^2 + 5p + 6 = \text{there is no GCF}$$

The magic number is  $\frac{5}{2}$ . We work out  $\left(p + \frac{5}{2}\right)^2$  on the margin:

$$\left(p + \frac{5}{2}\right)^2 = \left(p + \frac{5}{2}\right) \left(p + \frac{5}{2}\right) = p^2 + \frac{5}{2}p + \frac{5}{2}p + \frac{25}{4} = p^2 + \frac{10}{2}p + \frac{25}{4} = p^2 + 5p + \frac{25}{4}$$

So we know to smuggle in the missing third term,  $\frac{25}{4}$ .

$$p^2 + 5p + 6 = \text{our helper line: } \left(p + \frac{5}{2}\right)^2 = p^2 + 5p + \frac{25}{4}$$

$$= \underbrace{p^2 + 5p + \frac{25}{4}} + \frac{25}{4} + 6$$

realize complete square

$$= \left(p + \frac{5}{2}\right)^2 - \frac{25}{4} + \frac{6 \cdot 4}{4}$$

re-write 6 as  $\frac{6}{1} = \frac{24}{4}$

$$= \left(p + \frac{5}{2}\right)^2 - \frac{25}{4} + \frac{24}{4}$$

combine like terms

$$= \left(p + \frac{5}{2}\right)^2 - \frac{1}{4}$$

re-write  $\frac{1}{4}$  as  $\left(\frac{1}{2}\right)^2$

$$= \left(p + \frac{5}{2}\right)^2 - \left(\frac{1}{2}\right)^2$$

factor via the difference of squares theorem

$$= \left(p + \frac{5}{2} + \frac{1}{2}\right) \left(p + \frac{5}{2} - \frac{1}{2}\right)$$

combine like terms

$$= \left(p + \frac{6}{2}\right) \left(p + \frac{4}{2}\right)$$

simplify fractions

$$= \boxed{(p + 3)(p + 2)}$$

We check by multiplication:

$$(p + 3)(p + 2) = p^2 + 2p + 3p + 6 = p^2 + 5p + 6$$

Thus our result,  $(p + 3)(p + 2)$  is correct.

Note: These polynomials can easily be factored using trial and error and other methods. Why should we use completing the square? All other methods will break down once the numbers are less friendly. Then ONLY completing the square will work. Since those computations will be more difficult, you should learn this method while numbers are easy.

**Example 11.** Factor  $3x^2 - 4x - 319$  by completing the square.

If the leading coefficient is not 1, we will factor it out before completing the square.

$$3x^2 - 4x - 319 = 3 \left( x^2 - \frac{4}{3}x - \frac{319}{3} \right)$$

Half of the linear coefficient is  $-\frac{4}{3} \div 2 = -\frac{4}{3} \cdot \frac{1}{2} = -\frac{4}{6} = -\frac{2}{3}$ , thus we work out  $\left(x - \frac{2}{3}\right)^2$ .

$$\left(x - \frac{2}{3}\right)^2 = \left(x - \frac{2}{3}\right) \left(x - \frac{2}{3}\right) = x^2 - \frac{2}{3}x - \frac{2}{3}x + \frac{4}{9} = x^2 - \frac{4}{3}x + \frac{4}{9}$$

Thus we smuggle in  $\frac{4}{9}$ . The computation:

$$\begin{aligned} 3x^2 - 4x - 319 &= \\ &= 3 \left( x^2 - \frac{4}{3}x - \frac{319}{3} \right) & \left(x - \frac{2}{3}\right)^2 &= x^2 - \frac{4}{3}x + \frac{4}{9} \\ &= 3 \left( \underbrace{x^2 - \frac{4}{3}x + \frac{4}{9} - \frac{4}{9}}_{\left(x - \frac{2}{3}\right)^2} - \frac{319}{3} + \frac{4}{9} \right) \\ &= 3 \left( \left(x - \frac{2}{3}\right)^2 - \frac{4}{9} - \frac{957}{9} \right) \\ &= 3 \left( \left(x - \frac{2}{3}\right)^2 - \frac{961}{9} \right) & \frac{961}{9} &= \left(\frac{31}{3}\right)^2 \\ &= 3 \left( \left(x - \frac{2}{3}\right)^2 - \left(\frac{31}{3}\right)^2 \right) \\ &= 3 \left( x - \frac{2}{3} + \frac{31}{3} \right) \left( x - \frac{2}{3} - \frac{31}{3} \right) \\ &= 3 \left( x + \frac{29}{3} \right) \left( x - \frac{33}{3} \right) \\ &= 3 \left( x + \frac{29}{3} \right) (x - 11) = \boxed{(3x + 29)(x - 11)} \end{aligned}$$

We check:  $(3x + 29)(x - 11) = 3x^2 - 33x + 29x - 319 = 3x^2 - 4x - 319$ . Thus our answer is correct.

**Example 12.** Factor  $11x + 6x^2 - 10$  by completing the square.

We first rearrange the terms by degree and then factor out the leading coefficient.

$$11x + 6x^2 - 10 = 6x^2 + 11x - 10 = 6 \left( x^2 + \frac{11}{6}x - \frac{5}{3} \right)$$

Half of the linear coefficient is  $\frac{11}{6} \div 2 = \frac{11}{6} \left( \frac{1}{2} \right) = \frac{11}{12}$  and so we FOIL  $\left( x + \frac{11}{12} \right)^2$ .

$$\left( x + \frac{11}{12} \right)^2 = \left( x + \frac{11}{12} \right) \left( x + \frac{11}{12} \right) = x^2 + \frac{11}{12}x + \frac{11}{12}x + \left( \frac{11}{12} \right)^2 = x^2 + \frac{11}{6}x + \frac{121}{144} \qquad \frac{11}{12} + \frac{11}{12} = \frac{22}{12} = \frac{11}{6}$$

Thus we know to smuggle in  $\frac{121}{144}$

$$6 \left( x^2 + \frac{11}{6}x - \frac{5}{3} \right) =$$

$$= 6 \left( x^2 + \frac{11}{6}x + \frac{121}{144} - \frac{121}{144} - \frac{5}{3} \right)$$

$$\frac{5}{3} = \frac{5 \cdot 48}{3 \cdot 48} = \frac{240}{144}$$

$$= 6 \left( \left( x + \frac{11}{12} \right)^2 - \frac{121}{144} - \frac{240}{144} \right)$$

$$= 6 \left( \left( x + \frac{11}{12} \right)^2 - \frac{361}{144} \right)$$

$$\sqrt{361} = 19 \quad \text{and} \quad \sqrt{144} = 12$$

$$= 6 \left( \left( x + \frac{11}{12} \right)^2 - \left( \frac{19}{12} \right)^2 \right)$$

$$= 6 \left( x + \frac{11}{12} + \frac{19}{12} \right) \left( x + \frac{11}{12} - \frac{19}{12} \right)$$

$$= 6 \left( x + \frac{30}{12} \right) \left( x - \frac{8}{12} \right)$$

$$= 6 \left( x + \frac{5}{2} \right) \left( x - \frac{2}{3} \right) = 2 \left( x + \frac{5}{2} \right) 3 \left( x - \frac{2}{3} \right) = \boxed{(2x + 5)(3x - 2)}$$

We check:  $(2x + 5)(3x - 2) = 6x^2 - 4x + 15x - 10 = 6x^2 + 11x - 10$ . Thus our answer,  $(2x + 5)(3x - 2)$  is correct.



## Practice Problems

Completely factor each of the following by completing the square.

- |                     |                     |                            |                        |
|---------------------|---------------------|----------------------------|------------------------|
| 1. $x + x^2 - 12$   | 4. $x^2 - 17x + 72$ | 8. $112x + 2x^2 - 2x^3$    | 12. $6x^2 - 7x - 3$    |
| 2. $x^2 - x - 90$   | 5. $3x + 3x^2 - 60$ | 9. $5a^2 - 14a - 3$        | 13. $10p^2 - 11p - 6$  |
| 3. $11x + x^2 + 30$ | 6. $2x^2 - 6x + 17$ | 10. $33c^4 - 270c^3 - c^5$ | 14. $15x^2 - 34x + 15$ |
|                     | 7. $x + 2x^2 - 1$   | 11. $m + 6m^2 - 2$         |                        |



## Enrichment

Recall the difference of cubes theorem: For any quantities  $A$  and  $B$ ,  $A^3 - B^3$  can be factored as:

$$A^3 - B^3 = (A - B)(A^2 + AB + B^2).$$

1. Verify this identity by expanding the right-hand side.
2. Expand the given products. What do you notice?

$$(3x - 2)(5x + 1) \text{ and } (3A - 2B)(5A + B)$$

3. Consider now the second, quadratic factor in  $A^3 - B^3 = (A - B)(A^2 + AB + B^2)$ . Suppose we treat  $A$  as if it was  $x$ , and  $B$  as if it was 1. What happens when we try to factor  $A^2 + AB + B^2$  by completing the square?
4. What conclusions, if any, can we draw from our result?

## Part 4 - Irrational Solutions

Until now,  $x^2 - 4$  was factorable over the integers since  $x^2 - 4 = (x + 2)(x - 2)$ . Consider now  $x^2 - 5$ . This expression cannot be factored over the integers but can be factored over the reals as  $x^2 - 5 = x^2 - (\sqrt{5})^2 = (x + \sqrt{5})(x - \sqrt{5})$

One important achievement of our studies is the ability to solve quadratic equations with any real coefficients. In what follows, we will master this important skill. As a matter of fact, we can apply the method of completing the square to any quadratic expression. The steps are still the same - it's just that each step takes a bit more work.

We will now start interpreting numbers that are not perfect squares as squares anyway. For example, we can interpret 7 as  $(\sqrt{7})^2$ . This will be useful when we want to apply the difference of squares theorem. The expression  $x^2 - 4$  is factorable over the integers since  $x^2 - 4 = (x + 2)(x - 2)$ . On the other hand,  $x^2 - 7$  is not factorable over the integers. Over the real numbers,  $x^2 - 7$  can be factored as  $x^2 - 7 = x^2 - (\sqrt{7})^2 = (x + \sqrt{7})(x - \sqrt{7})$ .

**Example 13.** Solve the equation  $x^2 + 13 = 8x$  over the real numbers. Make sure to check your solutions.

**Solution:** Since the equation is quadratic, we reduce one side to zero and factor the other side by completing the square.

$$\begin{aligned} x^2 + 13 &= 8x \\ x^2 - 8x + 13 &= 0 & (x - 4)^2 &= x^2 - 8x + 16 \quad \text{so we smuggle in 16} \\ \underbrace{x^2 - 8x + 16}_{(x - 4)^2} - 16 + 13 &= 0 \\ (x - 4)^2 - 3 &= 0 & 3 &= (\sqrt{3})^2 \\ (x - 4)^2 - (\sqrt{3})^2 &= 0 & \text{factor via difference of squares theorem} \\ (x - 4 + \sqrt{3})(x - 4 - \sqrt{3}) &= 0 \end{aligned}$$

We apply the zero product rule, and solve for the zeroes of each linear factor.

$$\begin{aligned}
 x - 4 + \sqrt{3} &= 0 & \text{or} & & x - 4 - \sqrt{3} &= 0 \\
 x + \sqrt{3} &= 4 & & & x - \sqrt{3} &= 4 \\
 x_1 &= 4 - \sqrt{3} & & & x_2 &= 4 + \sqrt{3} \\
 x_1 &= 4 - \sqrt{3} & x_2 &= 4 + \sqrt{3} & \text{or in shorter notation: } & x_{1,2} = 4 \pm \sqrt{3}
 \end{aligned}$$

We check: if  $x = 4 - \sqrt{3}$ , then

$$\begin{aligned}
 \text{LHS} &= (4 - \sqrt{3})^2 + 13 = 16 - 4\sqrt{3} - 4\sqrt{3} + 3 + 13 = 32 - 8\sqrt{3} \\
 \text{RHS} &= 8(4 - \sqrt{3}) = 32 - 8\sqrt{3}
 \end{aligned}$$

If  $x = 4 + \sqrt{3}$ , then

$$\begin{aligned}
 \text{LHS} &= (4 + \sqrt{3})^2 + 13 = 16 + 4\sqrt{3} + 4\sqrt{3} + 3 + 13 = 32 + 8\sqrt{3} \\
 \text{RHS} &= 8(4 + \sqrt{3}) = 32 + 8\sqrt{3}
 \end{aligned}$$

Thus our solution,  $\boxed{4 + \sqrt{3} \text{ and } 4 - \sqrt{3}}$  is correct.

Our solutions are irrational numbers. When a quadratic equation has irrational solutions, all other methods break down, and only completing the square works. If the reader is familiar with methods such as grouping or the AC-method, or trial and error, try solving  $x^2 + 13 = 8x$  using any of those methods!

Being able to handle irrational solutions is a great advantage of the method of completing the square.

Please also note that in case of irrational solutions, writing the exact value necessitates the use of radicals. Recall that the exact value of an irrational number can not be captured as a fraction or as a terminating or repeating decimal, so radical form is our only option.

**Example 14.** Solve the equation  $x^2 = 4x + 1$  over the real numbers. Make sure to check your solutions.

**Solution:** We complete the square.

$$\begin{aligned}
 x^2 &= 4x + 1 \\
 x^2 - 4x - 1 &= 0 & (x - 2)^2 &= x^2 - 4x + 4 \\
 \underbrace{x^2 - 4x + 4}_{(x - 2)^2} - 4 - 1 &= 0 & 5 &= (\sqrt{5})^2 \\
 (x - 2)^2 - 5 &= 0 \\
 (x - 2)^2 - (\sqrt{5})^2 &= 0 \\
 (x - 2 + \sqrt{5})(x - 2 - \sqrt{5}) &= 0
 \end{aligned}$$

We apply the zero product rule to the product of the two linear factors.

$$\begin{array}{lll} x - 2 + \sqrt{5} & = & 0 \quad \text{or} \quad x - 2 - \sqrt{5} = 0 \\ x + \sqrt{5} & = & 2 \quad \quad \quad x - \sqrt{5} = 2 \\ x_1 & = & 2 - \sqrt{5} \quad \quad \quad x_2 = 2 + \sqrt{5} \end{array}$$

$$x_1 = 2 - \sqrt{5} \quad \text{and} \quad x_2 = 2 + \sqrt{5} \quad \text{or in shorter notation: } x_{1,2} = 2 \pm \sqrt{5}$$

We check: if  $x = 2 - \sqrt{5}$ , then

$$\begin{aligned} \text{LHS} &= (2 - \sqrt{5})^2 = 4 - 2\sqrt{5} - 2\sqrt{5} + 5 = 9 - 4\sqrt{5} \\ \text{RHS} &= 4(2 - \sqrt{5}) + 1 = 8 - 4\sqrt{5} + 1 = 9 - 4\sqrt{5} \end{aligned}$$

If  $x = 2 + \sqrt{5}$ , then

$$\begin{aligned} \text{LHS} &= (2 + \sqrt{5})^2 = 4 + 2\sqrt{5} + 2\sqrt{5} + 5 = 9 + 4\sqrt{5} \\ \text{RHS} &= 4(2 + \sqrt{5}) + 1 = 8 + 4\sqrt{5} + 1 = 9 + 4\sqrt{5} \end{aligned}$$

Thus our solution,  $\boxed{2 + \sqrt{5} \text{ and } 2 - \sqrt{5}}$  is correct.

**Example 15.** Solve the equation  $2x^2 + 10x + 9 = 0$  over the real numbers. Make sure to check your solutions.

**Solution:** We will factor by completing the square. We first factor out the leading coefficient.

$$\begin{aligned} 2x^2 + 10x + 9 &= 0 \\ 2\left(x^2 + 5x + \frac{9}{2}\right) &= 0 \end{aligned}$$

Half of the linear coefficient is  $\frac{5}{2}$ . Thus the complete square is  $\left(x + \frac{5}{2}\right)^2$

$$\left(x + \frac{5}{2}\right)^2 = \left(x + \frac{5}{2}\right)\left(x + \frac{5}{2}\right) = x^2 + \frac{5}{2}x + \frac{5}{2}x + \frac{25}{4} = x^2 + 5x + \frac{25}{4}$$

So we smuggle in  $\frac{25}{4}$ .

$$\begin{aligned} 2\left(\underbrace{x^2 + 5x + \frac{25}{4}} - \frac{25}{4} + \frac{9}{2}\right) &= 0 & \frac{9}{2} &= \frac{18}{4} \\ 2\left(\left(x + \frac{5}{2}\right)^2 - \frac{25}{4} + \frac{18}{4}\right) &= 0 \\ 2\left(\left(x + \frac{5}{2}\right)^2 - \frac{7}{4}\right) &= 0 \end{aligned}$$

We next apply the difference of squares theorem.  $\frac{7}{4}$  is the square of a number - its own square root.

$$\begin{aligned} 2 \left( \left( x + \frac{5}{2} \right)^2 - \frac{7}{4} \right) &= 0 & \frac{7}{4} &= \left( \sqrt{\frac{7}{4}} \right)^2 = \left( \frac{\sqrt{7}}{\sqrt{4}} \right)^2 = \left( \frac{\sqrt{7}}{2} \right)^2 \\ 2 \left( \left( x + \frac{5}{2} \right)^2 - \left( \frac{\sqrt{7}}{2} \right)^2 \right) &= 0 \\ 2 \left( x + \frac{5}{2} + \frac{\sqrt{7}}{2} \right) \left( x + \frac{5}{2} - \frac{\sqrt{7}}{2} \right) &= 0 \end{aligned}$$

We solve for the zeroes of both linear factors:

$$\begin{aligned} x + \frac{5}{2} + \frac{\sqrt{7}}{2} &= 0 & \text{or} & & x + \frac{5}{2} - \frac{\sqrt{7}}{2} &= 0 \\ x + \frac{\sqrt{7}}{2} &= -\frac{5}{2} & & & x - \frac{\sqrt{7}}{2} &= -\frac{5}{2} \\ x_1 &= -\frac{5}{2} - \frac{\sqrt{7}}{2} = \frac{-5 - \sqrt{7}}{2} & & & x_2 &= -\frac{5}{2} + \frac{\sqrt{7}}{2} = \frac{-5 + \sqrt{7}}{2} \end{aligned}$$

So the answer is  $x_1 = \frac{-5 - \sqrt{7}}{2}$  and  $x_2 = \frac{-5 + \sqrt{7}}{2}$  or, in a more compact form,  $x_{1,2} = \frac{-5 \pm \sqrt{7}}{2}$ .

We check: if  $x = \frac{-5 - \sqrt{7}}{2}$ , then

$$\begin{aligned} \text{LHS} &= 2x^2 + 10x + 9 = 2 \left( \frac{-5 - \sqrt{7}}{2} \right)^2 + 10 \left( \frac{-5 - \sqrt{7}}{2} \right) + 9 = \frac{2}{1} \cdot \frac{(-5 - \sqrt{7})^2}{2^2} + \frac{10}{1} \cdot \frac{-5 - \sqrt{7}}{2} + 9 \\ &= \frac{2(-5 - \sqrt{7})^2}{4} + \frac{10(-5 - \sqrt{7})}{2} + \frac{9}{1} = \frac{(-5 - \sqrt{7})^2}{2} + \frac{10(-5 - \sqrt{7})}{2} + \frac{18}{2} = \\ &= \frac{25 + 7 + 10\sqrt{7} - 50 - 10\sqrt{7} + 18}{2} = \frac{32 + 10\sqrt{7} - 50 - 10\sqrt{7} + 18}{2} = 0 = \text{RHS} \end{aligned}$$

and if  $x = \frac{-5 + \sqrt{7}}{2}$ , then

$$\begin{aligned} \text{LHS} &= 2x^2 + 10x + 9 = 2 \left( \frac{-5 + \sqrt{7}}{2} \right)^2 + 10 \left( \frac{-5 + \sqrt{7}}{2} \right) + 9 = \frac{2}{1} \cdot \frac{(-5 + \sqrt{7})^2}{2^2} + \frac{10}{1} \cdot \frac{-5 + \sqrt{7}}{2} + 9 \\ &= \frac{2(-5 + \sqrt{7})^2}{4} + \frac{10(-5 + \sqrt{7})}{2} + \frac{9}{1} = \frac{(-5 + \sqrt{7})^2}{2} + \frac{10(-5 + \sqrt{7})}{2} + \frac{18}{2} = \\ &= \frac{25 + 7 - 10\sqrt{7} - 50 + 10\sqrt{7} + 18}{2} = \frac{32 - 10\sqrt{7} - 50 + 10\sqrt{7} + 18}{2} = 0 = \text{RHS} \end{aligned}$$

Thus our solution,  $\frac{-5 - \sqrt{7}}{2}$  and  $\frac{-5 + \sqrt{7}}{2}$  is correct.

**Example 16.** Is there a real number that is exactly one less than its own square?

**Solution:** Let us denote this number by  $x$ . The square of  $x$  is then  $x^2$ . We set up and solve the equation comparing the number to its own square.

$$x = x^2 - 1$$

Since the equation is quadratic, we reduce one side to zero and factor the other side.

$$\begin{aligned} x &= x^2 - 1 \\ 0 &= x^2 - x - 1 \\ x^2 - x - 1 &= 0 & \left(x - \frac{1}{2}\right)^2 &= x^2 - x + \frac{1}{4} \\ \underbrace{x^2 - x + \frac{1}{4}} - \frac{1}{4} - 1 &= 0 \\ \left(x - \frac{1}{2}\right)^2 - \frac{1}{4} - \frac{4}{4} &= 0 \\ \left(x - \frac{1}{2}\right)^2 - \frac{5}{4} &= 0 & \frac{5}{4} &= \left(\sqrt{\frac{5}{4}}\right)^2 = \left(\frac{\sqrt{5}}{\sqrt{4}}\right)^2 = \left(\frac{\sqrt{5}}{2}\right)^2 \\ \left(x - \frac{1}{2}\right)^2 - \left(\frac{\sqrt{5}}{2}\right)^2 &= 0 \\ \left(x - \frac{1}{2} + \frac{\sqrt{5}}{2}\right) \left(x - \frac{1}{2} - \frac{\sqrt{5}}{2}\right) &= 0 \end{aligned}$$

$$\left(x - \frac{1}{2} + \frac{\sqrt{5}}{2}\right) \left(x - \frac{1}{2} - \frac{\sqrt{5}}{2}\right) = 0$$

$$\begin{aligned} \text{Either } x - \frac{1}{2} + \frac{\sqrt{5}}{2} &= 0 & \text{or } x - \frac{1}{2} - \frac{\sqrt{5}}{2} &= 0 \\ x + \frac{\sqrt{5}}{2} &= \frac{1}{2} & x - \frac{\sqrt{5}}{2} &= \frac{1}{2} \\ x_1 &= \frac{1}{2} - \frac{\sqrt{5}}{2} & x_2 &= \frac{1}{2} + \frac{\sqrt{5}}{2} \end{aligned}$$

The greater number,  $x_2 = \frac{1 + \sqrt{5}}{2} \approx 1.618034$  is a famous number. It is called the **golden mean** or **golden ratio**.

$\frac{1 + \sqrt{5}}{2}$  is often denoted by  $\varphi$  and has very interesting properties. For example, it is one less than its own square.

$$\begin{aligned} \varphi^2 - 1 &= \left(\frac{1 + \sqrt{5}}{2}\right)^2 - 1 = \frac{(\sqrt{5} + 1)^2}{2^2} - 1 = \frac{5 + 1 + 2\sqrt{5}}{4} - 1 = \frac{6 + 2\sqrt{5}}{4} - 1 = \frac{2(3 + \sqrt{5})}{2 \cdot 2} - 1 \\ &= \frac{3 + \sqrt{5}}{2} - \frac{2}{2} = \frac{3 + \sqrt{5}}{2} = \varphi \quad \checkmark \quad \text{and so } \varphi = \frac{1 + \sqrt{5}}{2} \text{ works.} \end{aligned}$$

Let us check the other number. Let us denote  $\frac{1 - \sqrt{5}}{2}$  by  $\psi$ .

$$\begin{aligned}\psi^2 - 1 &= \left(\frac{1 - \sqrt{5}}{2}\right)^2 - 1 = \frac{(\sqrt{5} - 1)^2}{2^2} - 1 = \frac{5 + 1 - 2\sqrt{5}}{4} - 1 = \frac{6 - 2\sqrt{5}}{4} - 1 = \frac{2(3 - \sqrt{5})}{2 \cdot 2} - 1 \\ &= \frac{3 - \sqrt{5}}{2} - \frac{2}{2} = \frac{3 - \sqrt{5}}{2} = \psi \quad \checkmark \quad \text{and so } \psi = \frac{1 - \sqrt{5}}{2} \text{ also works.}\end{aligned}$$

**Example 17.** Is there a real number that is exactly one less than its own square?

**Solution:** We factor out the leading coefficient and factor by completing the square.

$$\begin{aligned}3x^2 - 5x - 7 &= 0 \\ 3\left(x^2 - \frac{5}{3}x - \frac{7}{3}\right) &= 0\end{aligned}$$

Half of the linear coefficient is  $\frac{5}{3} \div 2 = \frac{5}{3} \cdot \frac{1}{2} = \frac{5}{6}$ . Thus the complete square is  $\left(x - \frac{5}{6}\right)^2$

$$\left(x - \frac{5}{6}\right)^2 = \left(x - \frac{5}{6}\right)\left(x - \frac{5}{6}\right) = x^2 - \frac{5}{6}x - \frac{5}{6}x + \frac{25}{36} = x^2 - \frac{5}{3}x + \frac{25}{36} \quad \text{So we smuggle in } \frac{25}{36}$$

$$\begin{aligned}3\left(\underbrace{x^2 - \frac{5}{3}x + \frac{25}{36}}_{\left(x - \frac{5}{6}\right)^2} - \frac{25}{36} - \frac{7}{3}\right) &= 0 \\ 3\left(\left(x - \frac{5}{6}\right)^2 - \frac{25}{36} - \frac{7 \cdot 12}{3 \cdot 12}\right) &= 0 \\ 3\left(\left(x - \frac{5}{6}\right)^2 - \frac{25}{36} - \frac{84}{36}\right) &= 0 \\ 3\left(\left(x - \frac{5}{6}\right)^2 - \frac{109}{36}\right) &= 0\end{aligned}$$

We next apply the difference of squares theorem.  $\frac{109}{36}$  is the square of a number - its own square root.

$$\frac{109}{36} = \left(\sqrt{\frac{109}{36}}\right)^2 = \left(\frac{\sqrt{109}}{\sqrt{36}}\right)^2 = \left(\frac{\sqrt{109}}{6}\right)^2$$

$$\begin{aligned}3\left(\left(x - \frac{5}{6}\right)^2 - \left(\frac{\sqrt{109}}{6}\right)^2\right) &= 0 \\ 3\left(x - \frac{5}{6} + \frac{\sqrt{109}}{6}\right)\left(x - \frac{5}{6} - \frac{\sqrt{109}}{6}\right) &= 0\end{aligned}$$

We solve for the zeroes of both linear factors:

$$\begin{array}{lll}
 x - \frac{5}{6} + \frac{\sqrt{109}}{6} = 0 & \text{or} & x - \frac{5}{6} - \frac{\sqrt{109}}{6} = 0 \\
 x + \frac{\sqrt{109}}{6} = \frac{5}{6} & & x - \frac{\sqrt{109}}{6} = \frac{5}{6} \\
 x_1 = \frac{5}{6} - \frac{\sqrt{109}}{6} = \boxed{\frac{5 - \sqrt{109}}{6}} & & x_2 = \frac{5}{6} + \frac{\sqrt{109}}{6} = \boxed{\frac{5 + \sqrt{109}}{6}}
 \end{array}$$

We check: if  $x = \frac{5 - \sqrt{109}}{6}$ , then

$$\begin{aligned}
 \text{LHS} &= 3x^2 - 5x - 7 = 3 \left( \frac{5 - \sqrt{109}}{6} \right)^2 - 5 \left( \frac{5 - \sqrt{109}}{6} \right) - 7 \\
 &= \frac{3}{1} \cdot \frac{(5 - \sqrt{109})^2}{36} - \frac{5}{1} \left( \frac{5 - \sqrt{109}}{6} \right) - 7 = \\
 &= \frac{3(25 - 5\sqrt{109} - 5\sqrt{109} + 109)}{36} - \frac{5(5 - \sqrt{109})}{6} - 7 = \frac{134 - 10\sqrt{109}}{12} - \frac{25 - 5\sqrt{109}}{6} - 7 \\
 &= \frac{134 - 10\sqrt{109}}{12} - \frac{2(25 - 5\sqrt{109})}{12} - 7 = \frac{134 - 10\sqrt{109} - 2(25 - 5\sqrt{109})}{12} - 7 \\
 &= \frac{134 - 10\sqrt{109} - 50 + 10\sqrt{109}}{12} - 7 = \frac{84}{12} - 7 = 7 - 7 = 0 = \text{RHS}
 \end{aligned}$$

and if  $x = \frac{5 + \sqrt{109}}{6}$ , then

$$\begin{aligned}
 \text{LHS} &= 3x^2 - 5x - 7 = 3 \left( \frac{5 + \sqrt{109}}{6} \right)^2 - 5 \left( \frac{5 + \sqrt{109}}{6} \right) - 7 \\
 &= \frac{3}{1} \cdot \frac{(5 + \sqrt{109})^2}{36} - \frac{5}{1} \left( \frac{5 + \sqrt{109}}{6} \right) - 7 = \\
 &= \frac{3(25 + 5\sqrt{109} + 5\sqrt{109} + 109)}{36} - \frac{5(5 + \sqrt{109})}{6} - 7 = \frac{134 + 10\sqrt{109}}{12} - \frac{25 + 5\sqrt{109}}{6} - 7 \\
 &= \frac{134 + 10\sqrt{109}}{12} - \frac{2(25 + 5\sqrt{109})}{12} - 7 = \frac{134 + 10\sqrt{109} - 2(25 + 5\sqrt{109})}{12} - 7 \\
 &= \frac{134 + 10\sqrt{109} - 50 - 10\sqrt{109}}{12} - 7 = \frac{84}{12} - 7 = 7 - 7 = 0 = \text{RHS}
 \end{aligned}$$

Thus our solution,  $\frac{5 + \sqrt{109}}{6}$  and  $\frac{5 - \sqrt{109}}{6}$  is correct.

**Example 18.** Find all numbers that are exactly three less than their own square.

Solution: Let us denote this number by  $x$ . Then its square is  $x^2$  and

$$\begin{aligned}
 x &= x^2 - 3 && \text{reduce one side to zero} \\
 0 &= x^2 - x - 3 \\
 x^2 - x - 3 &= 0 && \left(x - \frac{1}{2}\right)^2 = x^2 - x + \frac{1}{4} \\
 \underbrace{x^2 - x + \frac{1}{4}} - \frac{1}{4} - 3 &= 0 && 3 = \frac{12}{4} \\
 \left(x - \frac{1}{2}\right)^2 - \frac{13}{4} &= 0 && \sqrt{\frac{13}{4}} = \frac{\sqrt{13}}{\sqrt{4}} = \frac{\sqrt{13}}{2} \\
 \left(x - \frac{1}{2}\right)^2 - \left(\frac{\sqrt{13}}{2}\right)^2 &= 0 \\
 \left(x - \frac{1}{2} + \frac{\sqrt{13}}{2}\right) \left(x - \frac{1}{2} - \frac{\sqrt{13}}{2}\right) &= 0
 \end{aligned}$$

$$\begin{aligned}
 \text{Either } x - \frac{1}{2} + \frac{\sqrt{13}}{2} &= 0 && \text{or} && x - \frac{1}{2} - \frac{\sqrt{13}}{2} = 0 \\
 x + \frac{\sqrt{13}}{2} &= \frac{1}{2} && && x - \frac{\sqrt{13}}{2} = \frac{1}{2} \\
 x_1 &= \frac{1}{2} - \frac{\sqrt{13}}{2} = \frac{1 - \sqrt{13}}{2} && && x_2 = \frac{1}{2} + \frac{\sqrt{13}}{2} = \frac{1 + \sqrt{13}}{2}
 \end{aligned}$$

We check: If  $x_1 = \frac{1 - \sqrt{13}}{2}$ , then

$$\begin{aligned}
 x_1^2 - 3 &= \left(\frac{1 - \sqrt{13}}{2}\right)^2 - 3 = \frac{(1 - \sqrt{13})^2}{2^2} - 3 = \frac{1 + 13 - 2\sqrt{13}}{4} - 3 = \frac{14 - 2\sqrt{13}}{4} - 3 = \frac{2(7 - \sqrt{13})}{4} - 3 \\
 &= \frac{7 - \sqrt{13}}{2} - \frac{6}{2} = \frac{1 - \sqrt{13}}{2} = x_1 \checkmark \quad \text{Thus } x_1 = \frac{1 - \sqrt{13}}{2} \text{ works.}
 \end{aligned}$$

Let us see about the other number. If  $x_2 = \frac{1 + \sqrt{13}}{2}$ , then

$$\begin{aligned}
 x_2^2 - 3 &= \left(\frac{1 + \sqrt{13}}{2}\right)^2 - 3 = \frac{(1 + \sqrt{13})^2}{2^2} - 3 = \frac{1 + 13 + 2\sqrt{13}}{4} - 3 = \frac{14 + 2\sqrt{13}}{4} - 3 = \frac{2(7 + \sqrt{13})}{4} - 3 \\
 &= \frac{7 + \sqrt{13}}{2} - \frac{6}{2} = \frac{1 + \sqrt{13}}{2} = x_2 \checkmark \quad \text{Thus } x_2 = \frac{1 + \sqrt{13}}{2} \text{ also works.}
 \end{aligned}$$

We found two numbers; they both work.



## Practice Problems

Solve each of the following equations over the real numbers. Make sure to check your solutions.

1.  $x^2 - 13 = 6x$
2.  $x^2 = 10x + 3$
3.  $x^2 + x - 1 = 0$
4.  $3x^2 - 11x + 5 = 0$
5.  $-2x^2 + 8x - 29 = 0$
6.  $5x^2 - 4x = 20$
7.  $x^2 + 8x - 10 = 0$
8.  $3x^2 + x - 1 = 0$
9.  $x^2 + x + 1 = 0$

10. Find all numbers with the following property: twice the number is two less than the square of the number.
11. Find all numbers with the following property: the number is four less than the square of the number.



## Answers

### Practice Problems - Part 1

1.  $(x - 3)(x - 7)$
2.  $(x - 2)(x - 4)$
3.  $(y + 15)(y + 7)$
4.  $(b + 5)(b - 9)$
5.  $(a + 17)(a - 3)$
6. can not be factored
7.  $(x - 1)(x - 3)$
8. can not be factored
9.  $(x + 24)(x - 18)$
10. can not be factored
11.  $(m - 18)(m - 24)$
12.  $(x - 15)(x - 35)$
13.  $(y + 25)(y - 15)$
14.  $(x - 12)(x - 28)$
15. can not be factored
16.  $(q + 6)(q - 8)$
17.  $(x - 9)^2$
18.  $(t + 51)(t - 87)$
19.  $(x - 10)(x - 36)$
20.  $(q + 56)(q - 42)$

### Practice Problems - Part 2

1.  $2(x + 5)(x - 3)$
2.  $5a(a - 3)(a + 17)$
3.  $3b^2(b^2 - 10b + 26)$
4.  $2(x + 27)(x - 11)$
5.  $6c(c - 1)(c - 3)$
6.  $-d(d^2 + 2d + 2)$
7.  $-(x + 24)(x - 18)$
8. can not be factored
9.  $5ac(b + 12)(b - 10)$
10.  $2y^3(y + 7)(y + 5)$
11.  $3x^2y(y + 12)(y - 6)$
12.  $-5x(x + 20)(x - 10)$

### Practice Problems - Part 3

1.  $(x + 4)(x - 3)$
2.  $(x + 9)(x - 10)$
3.  $(x + 6)(x + 5)$
4.  $(x - 8)(x - 9)$
5.  $3(x + 5)(x - 4)$
6. can not be factored
7.  $2(x + 1)\left(x - \frac{1}{2}\right) = (x + 1)(2x - 1)$
8.  $-2x(x + 7)(x - 8)$
9.  $5\left(a + \frac{1}{5}\right)(a - 3) = (5a + 1)(a - 3)$
10.  $-c^3(c - 15)(c - 18)$
11.  $6\left(m + \frac{2}{3}\right)\left(m - \frac{1}{2}\right) = (3m + 2)(2m - 1)$
12.  $6\left(x + \frac{1}{3}\right)\left(x - \frac{3}{2}\right) = (3x + 1)(2x - 3)$
13.  $10\left(p + \frac{2}{5}\right)\left(p - \frac{3}{2}\right) = (5p + 2)(2p - 3)$
14.  $15\left(x - \frac{3}{5}\right)\left(x - \frac{5}{3}\right) = (5x - 3)(3x - 5)$

## Practice Problems - Part 4

1.  $3 \pm \sqrt{22}$  2.  $5 \pm 2\sqrt{7}$  3.  $\frac{-1 \pm \sqrt{5}}{2}$  4.  $\frac{11 \pm \sqrt{61}}{6}$  5. no real solution 6.  $\frac{2 \pm 2\sqrt{26}}{5}$   
 7.  $-4 \pm \sqrt{26}$  8.  $\frac{-1 \pm \sqrt{13}}{6}$  9. no real solution 10.  $1 + \sqrt{3}$  and  $1 - \sqrt{3}$  11.  $\frac{1 + \sqrt{17}}{2}$  and  $\frac{1 - \sqrt{17}}{2}$

## Enrichment

If we treat  $A$  as our variable  $x$ , and  $B$  like the numbers in the previous example. Then  $A^2$  is quadratic,  $AB$  is linear, and  $B^2$  is the constant term. We proceed to complete the square. Half of the linear coefficient is  $\frac{1}{2}B$ .

So the complete square we need is  $\left(A + \frac{B}{2}\right)^2$

$$\left(A + \frac{B}{2}\right)^2 = A^2 + \frac{AB}{2} + \frac{AB}{2} + \frac{B^2}{4} = A^2 + AB + \frac{B^2}{4}$$

So we will smuggle in  $\frac{B^2}{4}$

$$\begin{aligned} A^2 + AB + B^2 &= \\ A^2 + AB + \frac{B^2}{4} - \frac{B^2}{4} + B^2 &= \\ \left(A + \frac{B}{2}\right)^2 - \frac{B^2}{4} + \frac{4B^2}{4} &= \left(A + \frac{B}{2}\right)^2 + \frac{3}{4}B^2 \\ &= \left(A + \frac{B}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}B\right)^2 \end{aligned}$$

Since the sum of two squares can not be factored, the expression  $A^2 + AB + B^2$  is irreducible, can not be factored.

Conclusions:

If the square of two quantities are the same, that does not mean that the quantities are the same. They also could be opposites.

$$\begin{aligned} A^2 &= B^2 \\ A^2 - B^2 &= 0 \\ (A + B)(A - B) &= 0 \implies A = B \text{ or } A = -B \end{aligned}$$

But if the cube of two quantities are the same, they must equal. This is an interesting difference.

$$\begin{aligned} A^3 &= B^3 \\ A^3 - B^3 &= 0 \\ (A - B)(A^2 + AB + B^2) &= 0 \implies A = B \end{aligned}$$

We have just proved that the second, longer factor  $A^2 + AB + B^2$  is a sum of two squares, and therefore is always positive. Therefore, this factor does not produce any solutions. So the only solution of  $A^3 = B^3$  is  $A = B$ , while the solutions of  $A^2 = B^2$  are  $A = \pm B$ .