Completing the square is a powerful and elegant factoring technique. This technique is fundamentally different from other factoring techniques such as grouping or trial and error. While those other techniques enable us to factor over the integers, completing the square enables us to factor over the real numbers. If we wanted to factor expressions over the real numbers and the numbers turn out to be irrational, then grouping and trial and error will not work. Completing the square always does.

Completing the square is not just a factoring technique. We will soon see that it is also a way to understand quadratic expressions.

Part 1 - When the Leading Coefficient is 1

Definition: A **complete square** is an algebraic expression in which a sum (or difference) is being squared. For example, $(x-2)^2$, $(2a+3)^2$, and $(a+b-c)^2$ are complete squares.

Example 1. Expand each of the given complete squares.

a) $(x-2)^2$ b) $(2a+3)^2$

Solution: a) $(x-2)^2 = (x-2)(x-2) = x^2 - 2x - 2x + 4 = \boxed{x^2 - 4x + 4}$

b) $(2a+3)^2 = (2a+3)(2a+3) = 4a^2 + 6a + 6a + 9 = \boxed{4a^2 + 12a + 9}$

Example 2. Factor $-4x + x^2 - 21$ by completing the square.

Solution: Step 1. We re-arrange the terms by decreasing order of degree.

$$-4x + x^2 - 21 = x^2 - 4x - 21$$

Step 2A. We obtain the "magic number", that is half of the linear coefficient. The linear coefficient is the number multiplying x, **sign included**. In our example, the magic number is $\dfrac{-4}{2} = -2$. We do not write this line in the main computation.

Step 2B. We place an x in front of the magic number, and square the expression we obtained. Work out this computation on the margin, not in the main computation.

$$(x-2)^2 = (x-2)(x-2) = x^2 - 2x - 2x + 4 = x^2 - 4x + 4$$

Step 2C. We write the "helper line" $(x-2)^2 = x^2 - 4x + 4$ in the upper right hand side of the paper. We will use it twice. Our computation so far looks like this:

$$-4x + x^2 - 21 = x^2 - 4x - 21 \qquad\qquad (x-2)^2 = x^2 - 4x \boxed{+4}$$

Step 3. The smuggling step. What we have achieved in Step 2, is to have found the only perfect square that begins with the same two terms, $x^2 - 4x$ as our expression to be factored. We can see that the last term, $+4$ is missing. We complete the square as follows.

Step 3A. Write down our expression with one modification: we leave a gap between the second and third terms.

$$x^2 - 4x \qquad\quad - 21$$

Step 3B. We add zero to the expression by adding and then immediately subtracting 4 into the gap.

$$x^2 - 4x \; + 4 - 4 - 21$$

Step 4. We have obtained five terms. We re-write the first three terms as a perfect square (the second time we used the helper line) and combine the last two terms.

$$x^2 - 4x - 21 = \underbrace{x^2 - 4x + 4} - 4 - 21 = (x-2)^2 - 25$$

Step 5. We re-write the last number as a square.

$$(x-2)^2 - 25 = (x-2)^2 - 5^2$$

Step 6. If applies, we factor via the difference of squares theorem.

$$(x-2)^2 - 5^2 = (x-2+5)(x-2-5)$$

Step 7. (Cleanup) We simplify the factors by combining like terms.

$$(x-2+5)(x-2-5) = \boxed{(x+3)(x-7)}$$

Step 8. We check our result by multiplication.

$$(x+3)(x-7) = x^2 - 7x + 3x + 21 = x^2 - 4x + 21$$

Thus our result, $(x+3)(x-7)$ is correct.

The entire computation should look like this:

$$\begin{aligned}
-4x + x^2 - 21 = & \\
= \; & x^2 - 4x - 21 \qquad\qquad (x-2)^2 = x^2 - 4x + \boxed{4} \\
= \; & \underbrace{x^2 - 4x + 4} - 4 - 21 \\
= \; & (x-2)^2 - 25 \\
= \; & (x-2)^2 - 5^2 \\
= \; & (x-2+5)(x-2-5) \\
= \; & \boxed{(x+3)(x-7)}
\end{aligned}$$

We check: $(x+3)(x-7) = x^2 - 7x + 3x - 21 = x^2 - 4x - 21$

Example 3. Factor $32 - 18x + x^2$ by completing the square.

Solution: Step 1. We re-arrange the terms by decreasing order of degree.

$$32 - 18x + x^2 = x^2 - 18x + 32$$

Step 2A. We obtain the "magic number", that is half of the linear coefficient. (The linear coefficient is the number multiplying x, sign included.) In our example, the magic number is $\dfrac{-18}{2} = -9$.

Step 2B. We place an x in front of the magic number, and square the expression we obtained. Do not write this computation in the main computation.

$$(x-9)^2 = (x-9)(x-9) = x^2 - 9x - 9x + 81 = x^2 - 18x + 81$$

Step 2C. We write the "helper line" $(x-9)^2 = x^2 - 18x + 81$ in the upper right hand side of the paper.

$$x^2 - 18x + 32 \qquad\qquad (x-9)^2 = x^2 - 18x + \boxed{81}$$

Step 3. (The smuggling step.) What we have achieved in Step 3, is to have found the only perfect square that begins with the same two terms, $x^2 - 18x$ as our expression to be factored. We can see that the last term, $+81$ is missing. We complete the square by writing down our expression with a gap between the second and third terms, and then adding zero to the expression by adding and then immediately subtracting 81 in the gap.

$$x^2 - 18x + 32 = x^2 - 18x \ {+ 81 - 81} + 32$$

Step 4. We obtained five terms. We re-write the first three terms as a perfect square and combine the last two terms.

$$x^2 - 18x + 32 = \underbrace{x^2 - 18x + 81} - 81 + 32 = (x-9)^2 - 49$$

Step 5. We re-write the last number as a square.

$$(x-9)^2 - 49 = (x-9)^2 - 7^2$$

Step 6. If applies, we factor via the difference of squares theorem.

$$(x-9)^2 - 7^2 = (x - 9 + 7)(x - 9 - 7)$$

Step 7. (Cleanup) We simplify the factors by combining like terms.

$$(x - 9 + 7)(x - 9 - 7) = (x-2)(x-16)$$

Step 8. We check back by multiplication. (See below.)
The entire computation should look like this:

$$\begin{aligned} 32 - 18x + x^2 &= \\ &= x^2 - 18x + 32 \qquad\qquad (x-9)^2 = x^2 - 18x \boxed{+81} \\ &= \underbrace{x^2 - 18x + 81} - 81 + 32 \\ &= (x-9)^2 - 49 \\ &= (x-9)^2 - 7^2 \\ &= (x-9+7)(x-9-7) \\ &= \boxed{(x-2)(x-16)} \end{aligned}$$

We check: $(x-2)(x-16) = x^2 - 2x - 16x + 32 = x^2 - 18x + 32$

Thus our result, $(x-2)(x-16)$ is correct.

Example 4. Factor $28x + x^2 - 1173$ by completing the square.

Solution: We first rearrange the terms by degrees.

$28x + x^2 - 1173 =$

$= x^2 + 28x - 1173$ half of the linear coefficient is $\frac{28}{2} = 14$

We work out $(x+14)^2 = x^2 + 28x + 196$ on the margin. So we know to smuggle in 196.

$$\begin{aligned}
&= x^2 + 28x - 1173 && (x+14)^2 = x^2 + 28x \boxed{+196} \\
&= \underbrace{x^2 + 28x + 196} - 196 - 1173 && \text{realize comple square, combine like terms} \\
&= (x+14)^2 - 1369 && \text{from calculator, } \sqrt{1369} = 37 \\
&= (x+14)^2 - 37^2 && \text{difference of squares theorem} \\
&= (x+14+37)(x+14-37) && \text{combine like terms} \\
&= \boxed{(x+51)(x-23)}
\end{aligned}$$

We check by multiplication: $(x+51)(x-23) = x^2 - 23x + 51x - 1173 = x^2 + 28x - 1173$

Thus our result, $(x+51)(x-23)$ is correct.

Example 5. Factor $17 - 2a + a^2$ by completing the square.

Solution: We first rearrange the terms by degree.

$17 - 2a + a^2 =$

$$\begin{aligned}
&= 17 - 2a + a^2 && \text{rearrange terms} \\
&= a^2 - 2a + 17 && \text{the "magic number" is } \frac{-2}{2} = -1
\end{aligned}$$

We work out $(a-1)^2 = a^2 - 2a + 1$ on the margin.

$$\begin{aligned}
&= a^2 - 2a + 17 && (a-1)^2 = a^2 - 2a \boxed{+1}\text{, so we smuggle in 1} \\
&= \underbrace{a^2 - 2a + 1} - 1 + 17 && \text{realize complete square, combine like terms} \\
&= (a-1)^2 + 16
\end{aligned}$$

We can not apply the difference of squares theorem, since 16 is added, not subtracted. **The sum of squares can not be factored**, and so the expression $a^2 - 2a + 17$ can not be factored.

Example 6. One number is 12 more than another. Find these numbers if their product is 640.

Solution: Let us denote the smaller number by x. Then the other number can be represented as $x+12$. The equation will express the product of these numbers.

$$x(x+12)=640$$

This is a quadratic equation. So we solve it as always: reduce one side to zero, factor the other side, and then apply the zero product rule. This time we will end up with a quadratic expression with three terms (also called trinomial). We will factor by completing the square.

$$\begin{aligned}
x(x+12) &= 640 & \quad (x+6)^2 = x^2+12x+36 \\
x^2+12x-640 &= 0 \\
x^2+12x+36-36-640 &= 0 \\
(x+6)^2-676 &= 0 & \quad \sqrt{676}=26 \\
(x+6)^2-26^2 &= 0 \\
(x+6+26)(x+6-26) &= 0 \\
(x+32)(x-20) &= 0
\end{aligned}$$

$$x_1=-32 \qquad x_2=20$$

Recall that x denotes the smaller of the two numbers and $x+12$ denotes the greater number.

Case 1. If $x=-32$, then the other number is $-32+12=-20$. So the smaller number is -32, and the greater number is -20, These differ by 12 and their product is $-32(-20)=640$. So, this pair of numbers works.

Case 2. If $x=20$, then the other number is $20+12=32$. So the smaller number is 20, and the greater number is 32, These differ by 12 and their product is $20\cdot 32=640$. So, this pair of numbers also works.

Therefore, our solution is that there are two pairs of numbers: $\boxed{-20 \text{ with } -32 \text{ and } 20 \text{ with } 32}$.

If a word problem boils down to a quadratic equation, it may have two solutions.

Example 7. The square of a number is number is 7 more than six times the number. Find this number.

Solution: Let us denote the number by x. The equation will express the relationship between the square of this number and six times the number.

$$\begin{aligned}
x^2 &= 6x+7 & \quad &\text{reduce one side to zero} \\
x^2-6x-7 &= 0 & \quad &\text{faction, } (x-3)^2 = x^2-6x+9 \\
\underbrace{x^2-6x+9}-9-7 &= 0 & \quad &\text{smuggle in 9} \\
(x-3)^2-16 &= 0 \\
(x-3)^2-4^2 &= 0 \\
(x-3+4)(x-3-4) &= 0 \\
(x+1)(x-7) &= 0
\end{aligned}$$

$$x_1=-1 \qquad x_2=7$$

We check both possible solutions If $x=-1$, then $x^2=1$ and that is 7 greter than six times the number, $6(-1)=-6$. So -1 works. If $x=7$, then $x^2=49$ and 49 is 7 greter than six times the number, $6(7)=42$. So 7 also works. Therefore, our solution is, that there are two such numbers: $\boxed{-1 \text{ and } 7}$.

Practice Problems

Factor each of the following by completing the square.

1. $x^2 - 10x + 21$
2. $x^2 - 6x + 8$
3. $22y + y^2 + 105$
4. $b^2 - 4b - 45$
5. $14a + a^2 - 51$
6. $b^2 - 10b + 26$
7. $3 + x^2 - 4x$
8. $d^2 + 2d + 2$
9. $6x + x^2 - 432$
10. $x^2 - 14x + 58$
11. $m^2 - 42m + 432$
12. $x^2 - 50x + 525$
13. $10y + y^2 - 375$
14. $x^2 - 40x + 336$
15. $x^2 - 6x + 25$
16. $q^2 - 2q - 48$
17. $x^2 - 18x + 81$
18. $t^2 - 36t - 4437$
19. $x^2 - 46x + 360$
20. $14q + q^2 - 2352$
21. A number is 8 less than another. Find these numbers if their product is 240.
22. Twice th opposite of a number is 3 less than the dquare of the number. Find this number.
23. Eight time the opposite of a number is 15 more than the square of the number. Find this number.

Answers

1. $(x-3)(x-7)$ 2. $(x-2)(x-4)$ 3. $(y+15)(y+7)$ 4. $(b+5)(b-9)$ 5. $(a+17)(a-3)$
6. can not be factored 7. $(x-1)(x-3)$ 8. can not be factored 9. $(x+24)(x-18)$
10. can not be factored 11. $(m-18)(m-24)$ 12. $(x-15)(x-35)$ 13. $(y+25)(y-15)$
14. $(x-12)(x-28)$ 15. can not be factored 16. $(q+6)(q-8)$ 17. $(x-9)^2$ 18. $(t+51)(t-87)$
19. $(x-10)(x-36)$ 20. $(q+56)(q-42)$ 21. -20 with -12 and 12 with 20 22. -3 and 1 23. -5 and -3

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