It is strongly recommended to use fractions and not decimals. The steps are identical as before (see Factoring by Completing the Square Part 1 and Part 2), the only complications are that we need to perform the same computations with fractions.

Example 1. Factor $54x - 6x^2 + 60$ by completing the square.

$$\begin{aligned} 54x - 6x^2 + 60 &= && \text{rearrange terms} \\ -6x^2 + 54x + 60 &= && \text{factor out } -6 \\ -6\left(x^2 - 9x - 10\right) &= && \text{the "magic number" is } \frac{-9}{2} = -\frac{9}{2} \end{aligned}$$

We work out $\left(x - \frac{9}{2}\right)^2$ on the margin:

$$\left(x - \frac{9}{2}\right)^2 = \left(x - \frac{9}{2}\right)\left(x - \frac{9}{2}\right) = x^2 - \frac{9}{2}x - \frac{9}{2}x + \frac{81}{4} = x^2 - \frac{18}{2}x + \frac{81}{4} = x^2 - 9x + \frac{81}{4}$$

So we know to smuggle in the missing third term, $\frac{81}{4}$.

$-6\left(x^2 - 9x - 10\right) =$ our helper line: $\left(x - \frac{9}{2}\right)^2 = x^2 - 9x + \frac{81}{4}$

$$\begin{aligned} &= -6\left(\underbrace{x^2 - 9x + \frac{81}{4}} - \frac{81}{4} - \frac{10 \cdot 4}{1 \cdot 4}\right) && \text{realize comple square, re-write 10 as } \frac{10}{1} = \frac{40}{4} \\ &= -6\left(\left(x - \frac{9}{2}\right)^2 - \frac{81}{4} - \frac{40}{4}\right) && \text{combine like terms} \\ &= -6\left(\left(x - \frac{9}{2}\right)^2 - \frac{121}{4}\right) && \text{re-write } \frac{121}{4} \text{ as } \left(\frac{11}{2}\right)^2 \\ &= -6\left(\left(x - \frac{9}{2}\right)^2 - \left(\frac{11}{2}\right)^2\right) && \text{factor via the difference of squares theorem} \\ &= -6\left(x - \frac{9}{2} + \frac{11}{2}\right)\left(x - \frac{9}{2} - \frac{11}{2}\right) && \text{combine like terms} \\ &= -6\left(x + \frac{2}{2}\right)\left(x - \frac{20}{2}\right) && \text{simplify fractions} \\ &= \boxed{-6(x+1)(x-10)} \end{aligned}$$

We check by multiplication:

$$-6(x+1)(x-10) = -6\left(x^2 - 10x + x - 10\right) = -6\left(x^2 - 9x - 10\right) = -6x^2 + 54x + 60$$

Thus our result, $-6(x+1)(x-10)$ is correct.

Example 2. Factor $5p+p^2+6$ by completing the square.

$$\begin{aligned} 5p+p^2+6 &= \quad \text{rearrange terms} \\ p^2+5p+6 &= \quad \text{there is no GCF} \end{aligned}$$

The magic number is $\frac{5}{2}$. We work out $\left(p+\frac{5}{2}\right)^2$ on the margin:

$$\left(p+\frac{5}{2}\right)^2 = \left(p+\frac{5}{2}\right)\left(p+\frac{5}{2}\right) = p^2+\frac{5}{2}p+\frac{5}{2}p+\frac{25}{4} = p^2+\frac{10}{2}p+\frac{25}{4} = p^2+5p+\frac{25}{4}$$

So we know to smuggle in the missing third term, $\frac{25}{4}$.

$p^2+5p+6=$ our helper line: $\left(p+\frac{5}{2}\right)^2 = p^2+5p+\frac{25}{4}$

$$\begin{aligned}
&= \underbrace{p^2+5p+\frac{25}{4}}-\frac{25}{4}+6 && \text{realize comple square} \\
&= \left(p+\frac{5}{2}\right)^2-\frac{25}{4}+\frac{6\cdot 4}{1\cdot 4} && \text{re-write } 6 \text{ as } \frac{6}{1}=\frac{24}{4} \\
&= \left(p+\frac{5}{2}\right)^2-\frac{25}{4}+\frac{24}{4} && \text{combine like terms} \\
&= \left(p+\frac{5}{2}\right)^2-\frac{1}{4} && \text{re-write } \frac{1}{4} \text{ as } \left(\frac{1}{2}\right)^2 \\
&= \left(p+\frac{5}{2}\right)^2-\left(\frac{1}{2}\right)^2 && \text{factor via the difference of squares theorem} \\
&= \left(p+\frac{5}{2}+\frac{1}{2}\right)\left(p+\frac{5}{2}-\frac{1}{2}\right) && \text{combine like terms} \\
&= \left(p+\frac{6}{2}\right)\left(p+\frac{4}{2}\right) && \text{simplify fractions} \\
&= \boxed{(p+3)(p+2)}
\end{aligned}$$

We check by multiplication:

$$(p+3)(p+2) = p^2+2p+3p+6 = p^2+5p+6$$

Thus our result, $(p+3)(p+2)$ is correct.

Note: These polynomials can easily be factored using trial and error and other methods. Why should we use completing the square? All other methods will break down once the numbers are less friendly. Then ONLY completing the square will work. Since those computations will be more difficult, you should learn this method while numbers are easy.

Example 3. Factor $3x^2 - 4x - 319$ by completing the square.
If the leading coefficient is not 1, we will factor it out before completing the square.

$$3x^2 - 4x - 319 = 3\left(x^2 - \frac{4}{3}x - \frac{319}{3}\right)$$

Half of the linear coefficient is $-\frac{4}{3} \div 2 = -\frac{4}{3} \cdot \frac{1}{2} = -\frac{4}{6} = -\frac{2}{3}$, thus we work out $\left(x - \frac{2}{3}\right)^2$.

$$\left(x - \frac{2}{3}\right)^2 = \left(x - \frac{2}{3}\right)\left(x - \frac{2}{3}\right) = x^2 - \frac{2}{3}x - \frac{2}{3}x + \frac{4}{9} = x^2 - \frac{4}{3}x + \frac{4}{9}$$

Thus we smuggle in $\frac{4}{9}$. The computation:

$3x^2 - 4x - 319 =$

$$\begin{aligned}
&= 3\left(x^2 - \frac{4}{3}x - \frac{319}{3}\right) && \left(x - \frac{2}{3}\right)^2 = x^2 - \frac{4}{3}x + \frac{4}{9} \\
&= 3\left(\underbrace{x^2 - \frac{4}{3}x + \frac{4}{9}} - \frac{4}{9} - \frac{319 \cdot 3}{3 \cdot 3}\right) \\
&= 3\left(\left(x - \frac{2}{3}\right)^2 - \frac{4}{9} - \frac{957}{9}\right) \\
&= 3\left(\left(x - \frac{2}{3}\right)^2 - \frac{961}{9}\right) && \frac{961}{9} = \left(\frac{31}{3}\right)^2 \\
&= 3\left(\left(x - \frac{2}{3}\right)^2 - \left(\frac{31}{3}\right)^2\right) \\
&= 3\left(x - \frac{2}{3} + \frac{31}{3}\right)\left(x - \frac{2}{3} - \frac{31}{3}\right) \\
&= 3\left(x + \frac{29}{3}\right)\left(x - \frac{33}{3}\right) \\
&= 3\left(x + \frac{29}{3}\right)(x - 11) = \boxed{(3x + 29)(x - 11)}
\end{aligned}$$

We check: $(3x + 29)(x - 11) = 3x^2 - 33x + 29x - 319 = 3x^2 - 4x - 319$. Thus our answer is correct.

Example 4. Factor $11x+6x^2-10$ by completing the square.
We first rearrange the terms by degree and then factor out the leading coefficient.

$$11x+6x^2-10=6x^2+11x-10=6\left(x^2+\frac{11}{6}x-\frac{5}{3}\right)$$

Half of the linear coefficient is $\frac{11}{6}\div 2=\frac{11}{6}\left(\frac{1}{2}\right)=\frac{11}{12}$ and so we FOIL $\left(x+\frac{11}{12}\right)^2$.

$$\left(x+\frac{11}{12}\right)^2=\left(x+\frac{11}{12}\right)\left(x+\frac{11}{12}\right)=x^2+\frac{11}{12}x+\frac{11}{12}x+\left(\frac{11}{12}\right)^2=x^2+\frac{11}{6}x+\frac{121}{144}$$

$$\frac{11}{12}+\frac{11}{12}=\frac{22}{12}=\frac{11}{6}$$

Thus we know to smuggle in $\frac{121}{144}$

$$6\left(x^2+\frac{11}{6}x-\frac{5}{3}\right)=$$

$$\begin{aligned}
&= 6\left(\underbrace{x^2+\frac{11}{6}x+\frac{121}{144}}-\frac{121}{144}-\frac{5}{3}\right) \qquad \frac{5}{3}=\frac{5\cdot 48}{3\cdot 48}=\frac{240}{144}\\
&= 6\left(\left(x+\frac{11}{12}\right)^2-\frac{121}{144}-\frac{240}{144}\right)\\
&= 6\left(\left(x+\frac{11}{12}\right)^2-\frac{361}{144}\right) \qquad \sqrt{361}=19 \text{ and } \sqrt{144}=12\\
&= 6\left(\left(x+\frac{11}{12}\right)^2-\left(\frac{19}{12}\right)^2\right)\\
&= 6\left(x+\frac{11}{12}+\frac{19}{12}\right)\left(x+\frac{11}{12}-\frac{19}{12}\right)\\
&= 6\left(x+\frac{30}{12}\right)\left(x-\frac{8}{12}\right)\\
&= 6\left(x+\frac{5}{2}\right)\left(x-\frac{2}{3}\right)\\
&= 2\left(x+\frac{5}{2}\right)3\left(x-\frac{2}{3}\right)=\boxed{(2x+5)(3x-2)}
\end{aligned}$$

We check: $(2x+5)(3x-2)=6x^2-4x+15x-10=6x^2+11x-10$. Thus our answer, $(2x+5)(3x-2)$ is correct.

Practice Problems

Completely factor each of the following by completing the square.

1. $x+x^2-12$
2. x^2-x-90
3. $11x+x^2+30$
4. $x^2-17x+72$
5. $3x+3x^2-60$
6. $2x^2-6x+17$
7. $x+2x^2-1$
8. $112x+2x^2-2x^3$
9. $5a^2-14a-3$
10. $33c^4-270c^3-c^5$
11. $m+6m^2-2$
12. $6x^2-7x-3$
13. $10p^2-11p-6$
14. $15x^2-34x+15$

Answers

1. $(x+4)(x-3)$ 2. $(x+9)(x-10)$ 3. $(x+6)(x+5)$ 4. $(x-8)(x-9)$ 5. $3(x+5)(x-4)$

6. can not be factored 7. $2(x+1)\left(x-\frac{1}{2}\right)=(x+1)(2x-1)$ 8. $-2x(x+7)(x-8)$

9. $5\left(a+\frac{1}{5}\right)(a-3)=(5a+1)(a-3)$ 10. $-c^3(c-15)(c-18)$ 11. $6\left(m+\frac{2}{3}\right)\left(m-\frac{1}{2}\right)=(3m+2)(2m-1)$

12. $6\left(x+\frac{1}{3}\right)\left(x-\frac{3}{2}\right)=(3x+1)(2x-3)$ 13. $10\left(p+\frac{2}{5}\right)\left(p-\frac{3}{2}\right)=(5p+2)(2p-3)$

14. $15\left(x-\frac{3}{5}\right)\left(x-\frac{5}{3}\right)=(5x-3)(3x-5)$

