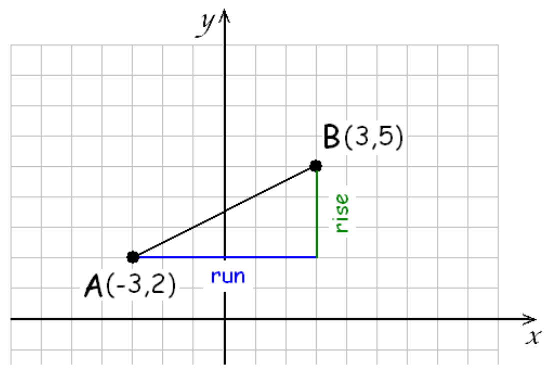
Until now, we focused on finding the graph of a given linear equation. In what follows, we will approach lines differently. Given some conditions of the line, *we* will come up with the equation. Let us first review a few things about straight lines.Let us first recall a definition.

Definition: Equations that are in x, or in y, or in x and y can be graphed. **The graph of such an equation is the set of all points $P(x, y)$ for which the coordinates x and y form a solution of the equation.**

If the equation is linear in x and y, the graph of the equation is a line. If one of the variables is missing, and the equation is linear in the other (such as $x = 2$ or $y = -3$), the graph of the equation is a line.

The slope of a line is a very important concept. It expresses the steepness of a line. Given two points on a line, the slope is a signed ratio expressing the vertical change per the horizontal change between the two points.

$$m = \frac{\text{rise}}{\text{run}} = \frac{\text{change in the } y - \text{coordinate}}{\text{change in the } x - \text{coordinate}} = \frac{y_2 - y_1}{x_2 - x_1}$$



Definition: The slope of a line (usually denoted by m) is a signed ratio expressing the steepness of the line. Given points $A(x_1, y_1)$ and $B(x_2, y_2)$, the slope of the line connecting these points is

$$m = \frac{\text{rise}}{\text{run}} = \frac{y_2 - y_1}{x_2 - x_1}$$

This formula is called the **slope formula**.

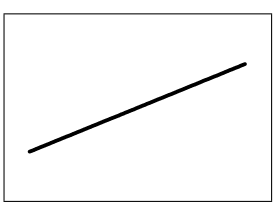
Example 1. Find the slope of the line determined by the points $A(-3, 2)$ and $B(3, 5)$.

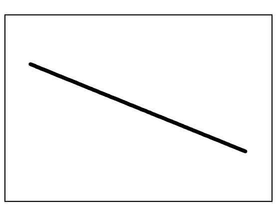
Solution: a) We find the slope determined by the points. $(-3, 2) = (x_1, y_1)$ and $(3, 5) = (x_2, y_2)$ using the slope formula.

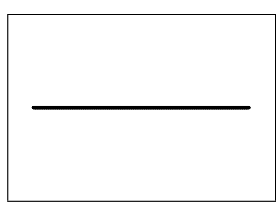
$$m = \frac{\text{rise}}{\text{run}} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{5 - 2}{3 - (-3)} = \frac{3}{6} = \frac{1}{2}$$

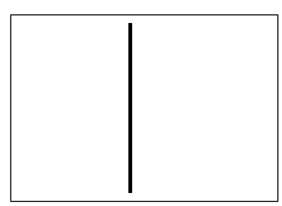
So the slope is $\boxed{\frac{1}{2}}$.

Note that **the slope of all horizontal lines is zero** and that **vertical lines have no slope**.


line with positive slope


line with negative slope


line with slope $m = 0$


line with no slope

When the equation of a line is given, we can find its slope by finding two points on the line and applying the slope formula. However, there are easier ways to find the slope of a line, given its equation.

Theorem: Suppose that the equation of a line is $y = mx + b$. Then the slope of this line is m, the coefficient of x in the equation, and the line's $y-$intercept is the point $(0, b)$. This form of an equation is called the **slope-intecept form**.

Proof: Suppose that $A(x_A, y_A)$ is a point on the line $y = mx + b$. Since this point is on the line, $y_A = mx_A + b$, and so the coordinates of this point is $(x_A, mx_A + b)$. Let us find another point B with an $x-$coordinate 1 greater than x_A. Then the new point, B has an $x-$coordinate $x_B = x_A + 1$, and the $y-$cooridnate is $y_B = mx_B + b = m(x_A + 1) + b = mx_A + m + b$. Applying the slope formula for A and B

$$\text{slope} = \frac{y_B - y_A}{x_B - x_A} = \frac{(mx_A + m + b) - (mx_A + b)}{(x_A + 1) - x_A} = \frac{mx_A + m + b - mx_A - b}{x_A + 1 - x_A} = \frac{m}{1} = m$$

If we substitute $x = 0$ into the equation of the line $y = mx + b$, the result is $y = m \cdot 0 + b = b$. Therefore, the $y-$intercept of this line is $(0, b)$. ■

The intercept-slope form of a line tells us a great deal about the line. For example, the line $y = 2x - 3$ has slope $m = 2$ and $y-$intercept $(0, -3)$. The line $y = -\frac{2}{3}x + 1$ has slope $m = -\frac{2}{3}$, and $y-$intercept $(0, 1)$. The line $y = x - 5$ has slope $m = 1$ and $y-$intercept $(0, -5)$. We have seen before that graphing a line is a matter of seconds if its equation is given in the slope-intercept form.

We need to be careful. If the line is given in a form other than the slope-intercept form, the coefficient of x is NOT the slope. For example, the slope of the line $2x - 3y = -12$ is NOT 2.

Definition: The equation of a line can also be given in the form of $Ax + By = C$, where A, B, and C are real numbers. This form of the equation is called the **general form**.

Example 2. Find the slope of the line $2x - 3y = -12$.

Solution: The slope of this line is not 2. We have to first transform the equation into the slope-intersept form by solving for y in terms of x.

$$\begin{aligned} 2x - 3y &= -12 && \text{add } 3y \text{ to both sides} \\ 2x &= 3y - 12 && \text{add } 12 \\ 2x + 12 &= 3y && \text{divide by 3 (also, swap the two sides)} \\ y &= \frac{2x + 12}{3} = \frac{2}{3}x + \frac{12}{3} = \frac{2}{3}x + 4 \end{aligned}$$

So we have the equation $y = \frac{2}{3}x + 4$. This equation is in the slope-intercept form, so the slope is the coefficient of x. Thus $m = \boxed{\frac{2}{3}}$.

A natural question is: which equation is better to use, the general form or the slope-intecept form. One possible answer is that they are equally useful, with different adventages. The slope-intecept form is excellent when graphing the line or finding geomteric properties of it. The general for better serves us when the focus is on algebrais computatons. For example, the general form might be preferred when finding the intersection of two lines.

Example 3. Write an equation for the line with slope $m = -\frac{1}{4}$ and $y-$intercept $(0, 3)$.

Solution: There is not much we need to do. Since the slope and the $y-$intercept was given, we immediately have that $m = -\frac{1}{4}$ and $b = 3$ in the equation $y = mx + b$. Thus the equation is $\boxed{y = -\frac{1}{4}x + 3}$.

There is one more form of equation of a line that is extremely useful, it is called the point-slope form.

Definition: Suppose that a line has slope m and passes through the point $P(x_P, y_P)$. **The point-slope form** of the equation of this line is

$$y - y_P = m(x - x_P)$$

Example 4. Write the point-slope form equation for each of the lines passing through the given point and having the given slope.

a) $P(-5, 1)$ and $m = -2$ b) $P(3, -2)$ and $m = -\frac{2}{3}$ c) $P(4, 0)$ and $m = 1$

Solution: a) We write the point-slope form. $m = -2$, and $P(-5, 1)$ means that $x_P = -5$ and $y_P = 1$.

$y - y_P = m(x - x_P)$ becomes $y - 1 = -2(x - (-5))$

We simplify the equation: $\boxed{y - 1 = -2(x + 5)}$.

b) We write the point-slope form. $m = -\frac{2}{3}$, and $P(3, -2)$ means that $x_P = 3$ and $y_P = -2$.

$y - y_P = m(x - x_P)$ becomes $y - (-2) = -\frac{2}{3}(x - 3)$

We simplify the equation: $\boxed{y + 2 = -\frac{2}{3}(x - 3)}$.

c) We write the point-slope form. $m = 1$, and $P(4, 0)$ means that $x_P = 4$ and $y_P = 0$.

$y - y_P = m(x - x_P)$ becomes $y - 0 = 1(x - 4)$

We simplify the equation: $\boxed{y = x - 4}$.

Theorem: Two lines are parallel if and only if they have the same slope or both lines are vertical.

Example 5. Write an equation for the line that passes through the point $(6, -2)$ and is parallel to the line $3x + 4y = 10$.

Solution: Let us first figure out the slope of this other, parallel line. We transforming the equation given to slope-intercept form and read the slope.

$$3x + 4y = 10 \qquad \text{subtract } 3x \text{ from both sides}$$

$$4y = -3x + 10 \qquad \text{divide by } 4$$

$$y = \frac{-3x + 10}{4} = -\frac{3}{4}x + \frac{10}{4} = -\frac{3}{4}x + \frac{5}{2}$$

If our line is to be parallel to this line, its slope must be the same. Now we can simply apply the point slope form with $m = -\frac{3}{4}$ and $P(6, -2)$.

$$y - y_P = m(x - x_P) \text{ becomes } y - (-2) = -\frac{3}{4}(x - 6)$$

We simplify the equation: $\boxed{y + 2 = -\frac{3}{4}(x - 6)}$

Recall that two lines are perpendicular if they intersect each other at an angle of 90° (also called right angle).

Theorem: Two lines are perpendicular if and only if their slopes m_1 and m_2 are negative reciprocals of each other, i.e. $m_1 m_2 = -1$, or one line is vertical and the other one is horizontal.

Example 6. Write an equation for the line that passes through the point $(6, -2)$ and is perpendicular to the line $3x + 4y = 10$.

Solution: This is the same line as in the previous example, so we know that its slope is $-\frac{3}{4}$. Our line is to be perpendicular to this line, so its slope needs to be the negative reciprocal of $-\frac{3}{4}$. That is $\frac{4}{3}$. We can check by multiplying the two slopes: the product must be -1. Now we simply apply the point slope form with $m = \frac{4}{3}$ and $P(6, -2)$.

$$y - y_P = m(x - x_P) \text{ becomes } y - (-2) = \frac{4}{3}(x - 6) \text{ or } \boxed{y + 2 = \frac{4}{3}(x - 6)}.$$

There are a number of interesting and important applications of our new skill of coming up with equations of lines based on certain geometric properties.

Example 7. Consider the triangle determined by $A(-4, 4)$, $B(1, 7)$, and $C(6, -1)$. Write an equation for the altitude drawn to side AC.

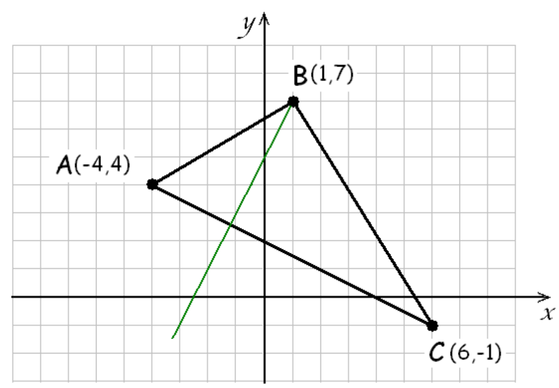
Solution: This task might seem intimidating at first, but it is not much different from our previous example. The height or altitude drawn to side AC is simply a line that is perpendicular to line AC and contains point B.

We first compute the slope of line segment AC. We use the slope formula with points $A(-4, 4)$ and $C(6, -1)$.

$$m_{AC} = \frac{y_C - y_A}{x_C - x_A} = \frac{-1 - 4}{6 - (-4)} = \frac{-5}{10} = -\frac{1}{2}$$

The height drawn to side AC is perpendicular to side AC and so its slope must be the negative reciprocal of m_{AC}. That is a slope of 2. Now we simply apply the point slope form with $m = 2$ and $B(1, 7)$.

$y - y_P = m(x - x_P)$ becomes $\boxed{y - 7 = 2(x - 1)}$.



If we are to present the equation in the slope-intercept from, we distribute the 2 on the right-hand side and then isolate y on the other side.

$$y - 7 = 2(x - 1)$$

$$y - 7 = 2x - 2$$

$$\boxed{y = 2x + 5}$$

Example 8. Find an equation for the line connecting the points $P(3,-8)$ and $Q(-2,2)$.

Solution: We will use the slope formula to find the slope of the line connecting P and Q. Then we apply the point-slope form with either P or Q.

$$m_{PQ}=\frac{y_Q-y_P}{x_Q-x_P}=\frac{2-(-8)}{-2-3}=\frac{10}{-5}=-2$$

Both $\boxed{y+8=-2(x-3)}$ and $\boxed{y-2=-2(x+2)}$ are correct. We can transform both equations to the slope intercept form.

$$\begin{aligned} y+8&=-2(x-3) \qquad & y-2&=-2(x+2)\\ y+8&=-2x+6 & y-2&=-2x-4\\ y&=-2x-2 & y&=-2x-2 \end{aligned}$$

So the two point-slope equations belong to the same line, whose slope-intercept form is $\boxed{y=-2x-2}$.

Theorem: Suppose that $A(x_A,y_A)$ and $B(x_B,y_B)$ are two points given. Let M denote the midpoint of line segment AB. The coordinates of $M(x_M,y_M)$ are the average of the corresponding coordinates of A and B.

$$x_M=\frac{x_A+x_B}{2} \text{ and } y_M=\frac{y_A+y_B}{2}$$

Example 9. Find the midpoint of the line segment AB where points A and B are given as $A(-5,4)$ and $B(3,-4)$.

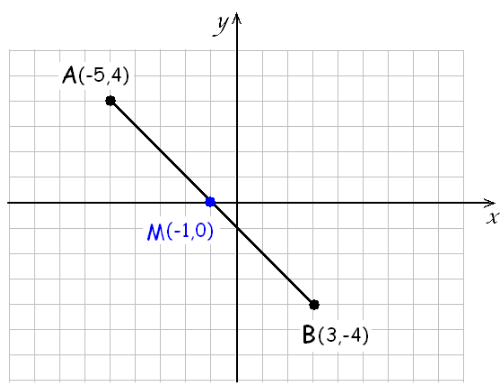
Solution: The $x-$coordinate of the midpoint is the average of the $x-$coordinates of A and B.

$$x_M=\frac{x_A+x_B}{2}=\frac{-5+3}{2}=\frac{-2}{2}=-1$$

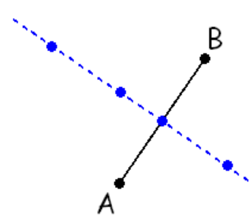
Similarly, the $y-$coordinate of the midpoint is the average of the $y-$coordinates of A and B.

$$y_M=\frac{y_A+y_B}{2}=\frac{4+(-4)}{2}=\frac{0}{2}=0$$

Thus the midpoint is $M(-1,0)$.



Definition: Suppose that points A and B are given. The **perpendicular bisector** of line segment AB is the set of all points P in the plane that are equidistant to points A and B.

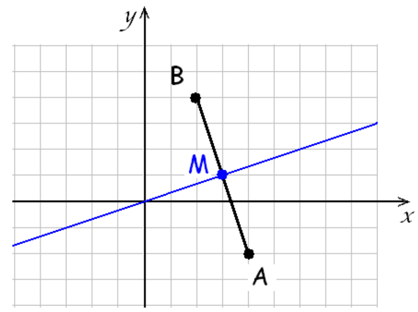


Although the definition is given in terms of distances, there are two properties of the perpendicular bisector that we use when finding their equation. **The perpendicular bisector of line segment AB is perpendicular to that line segment, and passes through the midpoint of the line segment.**

Example 10. Find an equation of the perpendicular bisector of the line segment AB where $A(4,-2)$ and $B(2,4)$.

Solution: The perpendicular bisector of the line segment passes through its midpoint M. We compute the coordinates of M.

$$x_M = \frac{x_A + x_B}{2} = \frac{4+2}{2} = 3 \text{ and}$$
$$y_M = \frac{y_A + y_B}{2} = \frac{-2+4}{2} = 1 \qquad \text{Thus} \quad M(3,1)$$



Next, we compute the slope of the line connecting A and B.

$$m_{AB} = \frac{y_B - y_A}{x_B - x_A} = \frac{4-(-2)}{2-4} = \frac{6}{-2} = -3$$

The slope of the perpendicular bisector is the negative reciprocal of m_{AB}, that is $\frac{1}{3}$. We are ready to write the equation in point-slope form, with $m = \frac{1}{3}$ and $M(3,1)$. Thus an equation of the perpendicular bisector is

$$\boxed{y - 1 = \frac{1}{3}(x-3)}$$

An Application

When we convert units, we usually deal with a simple ration problem. If 1 kilogram (kg) is 2.2 pounds (lb), then 2 kilograms are 4.4 pounds, 10 kilograms are 22 pounds, and so on. Conversion of temperature is an exception. In the USA, we use Farenheit, (°F) and in most of Europe, Celsius (°C) is used. But conversion between the two units is more complicated than a ratio problem.

Celsius was established as follows. The temperature at which water freezes was defined as 0 °C. The temperature at which water boils was defined as 100 °C. An increase of 1 °C would be the same as one-hundredth of the difference between freezing and boiling points.

We can always come up with the conversion formula between Celsius and Farenheit if we remember the two temperatures listed before, and if we look at it as the equation of a line, given two points on it.

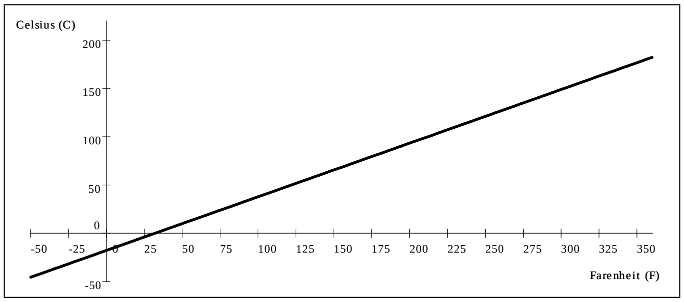
Suppose we want to convert from Farenheit to Celsius. Let us create a coordinate system in which the x−axis represents Farenheit, and the y−axis represents Celsius measurement. Water freezes at 32 °F, this temperature was defined to be 0 °C. Water boils at 212 °F, this temperature was defined to be 100 °C. Therefore the conversion formula would be the equation of the line connecting the points $(32,0)$ and $(212,100)$.

We compute the slope using the slope formula: $m = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{100-0}{212-32} = \dfrac{100}{180} = \dfrac{5}{9}$.

We now use the point-slope form with $(32,0)$ to write the equation of the line.

$$y - 0 = \frac{5}{9}(x-32) \qquad \Longrightarrow \qquad y = \frac{5}{9}(x-32)$$

This means that the conversion formula from Farenheit to Celsius is $C = \frac{5}{9}(F-32)$.



For example, to convert 77 Farenheit to Celsius, we write $C = \frac{5}{9}(77-32) = 25$. Therefore, 77 °F = 25 °C.

Sample Problems

1. In each case, find the slope of the line determined by the two points given.

   a) $(5,-1)$ and $(1,3)$ b) $(2,1)$ and $(6,3)$ c) $(8,-1)$ and $(-1,-1)$ d) $(7,3)$ and $(7,-4)$

2. For each straight line given, find its slope.

   a) $y=-2x+5$. b) $2x-5y=8$ c) $y=3$ d) $7(x-3)+x-2y=2y+1$

   e) $x=-\dfrac{1}{2}$ f) a line parallel to $y=3x-1$ g) a line perpendicular to $y=3x-1$

3. Find an equation for the straight line that has slope 4 and $y-$intercept $(0,-5)$.

4. Find an equation for the straight line that has slope $-\dfrac{1}{2}$ and passes through the point $(4,1)$.

5. Find an equation of the straight line that is parallel to $3x-2y=12$ and passes through the point $(8,-1)$.

6. Find an equation of the straight line that is perpendicular to $x+5y=-3$ and passes through the point $(-1,4)$.

7. Find an equation of the straight line that passes through the points $(3,-1)$ and $(1,5)$.

8. Find the midpoint of the line segment determined by $A(7,-4)$ and $B(3,-2)$.

9. Consider the triangle determined by $A(-5,-3)$, $B(-1,3)$, and $C(7,1)$.

   a) Find an equation for the line that connects A and B.

   b) Find the midpoint of line segment AB.

   c) Find an equation for the perpendicular bisector of side AB.

   d) Find an equation for the height (or altitude) belonging to side AC.

Practice Problems

1. In each case, find the slope of the line determined by the two points given.

   a) $(-2,-3)$ and $(3,-1)$ b) $(12,-7)$ and $(6,-3)$ c) $(2,5)$ and $(2,10)$ d) $(12,9)$ and $(-6,9)$

2. For each straight line given, find its slope.

   a) $y=\frac{3}{4}x$ b) $3x+10y=1$ c) $y=0$ d) $3(y-2)-2(5-x)=3x-4$

   e) $x=11$ f) a line parallel to $y=\frac{1}{2}x+7$ g) a line perpendicular to $y=\frac{1}{2}x+7$

3. Find an equation for the straight line that has slope -2 and $y-$intercept $(0,7)$.

4. Find an equation for the straight line that has slope $\frac{2}{3}$ and passes through the point $(-6,7)$.

5. Find an equation of the straight line that is parallel to $3x-2y=12$ and passes through the point $(8,-1)$.

6. Find an equation of the straight line that is perpendicular to $x+5y=-3$ and passes through the point $(-1,4)$.

7. Find an equation of the straight line that passes through the points $(2,10)$ and $(5,-2)$.

8. Find an equation of the straight line that passes through the points $(-4,5)$ and $(6,0)$.

9. Consider the triangle determined by $A(6,1)$, $B(0,7)$, and $C(-4,-3)$.

   a) Find an equation for the line that connects the given points.

      i) A and B ii) A and C iii) B and C

   b) Find the midpoint of each of the given line segments.

      i) AB ii) AC iii) BC

   c) Find an equation for the perpendicular bisector of each of the given line segments.

      i) AB ii) AC iii) BC

   d) Find an equation for the height (or altitude) belonging to each of the given sides.

      i) AB ii) AC iii) BC

Answers

Sample Problems

1. a) -1 b) $\dfrac{1}{2}$ c) 0 d) This line has no slope

2. a) -2 b) $\dfrac{2}{5}$ c) 0 d) 2 e) This line has no slope f) 3 g) $-\dfrac{1}{3}$

3. $y = 4x - 5$ 4. $y = -\dfrac{1}{2}x + 3$ 5. $y = \dfrac{3}{2}x - 13$ 6. $y = 5x + 9$ 7. $y = -3x + 8$

8. $(5, -3)$ 9. a) $y + 3 = \dfrac{3}{2}(x + 5)$ or $y - 3 = \dfrac{3}{2}(x + 1)$ b) $M(-2, 0)$ c) $y = -\dfrac{2}{3}x - 2$ d) $y = -3x$

Practice Problems

1. a) $\dfrac{2}{5}$ b) $-\dfrac{2}{3}$ c) This line has no slope d) 0

2. a) $\dfrac{3}{4}$ b) $-\dfrac{3}{10}$ c) 0 d) $\dfrac{1}{3}$ e) This line has no slope f) $\dfrac{1}{2}$ g) -2 3. $y = -2x + 7$

4. $y = \dfrac{2}{3}x + 11$ 5. $y = \dfrac{3}{2}x - 13$ 6. $y = 5x + 9$ 7. $y = -4x + 18$ 8. $y = -\dfrac{1}{2}x + 3$

9. a) i) $y = -x + 7$ ii) $y - 1 = \dfrac{2}{5}(x - 6)$ or $y = \dfrac{2}{5}x - \dfrac{7}{5}$ iii) $y = \dfrac{5}{2}x + 7$
   b) i) $(3, 4)$ ii) $(1, -1)$ iii) $(-2, 2)$
   c) i) $y = x + 1$ ii) $y + 1 = -\dfrac{5}{2}(x - 1)$ or $y = -\dfrac{5}{2}x + \dfrac{3}{2}$ iii) $y - 2 = -\dfrac{2}{5}(x + 2)$ or $y = -\dfrac{2}{5}x + \dfrac{6}{5}$
   d) i) $y = x + 1$ ii) $y = -\dfrac{5}{2}x + 7$ iii) $y - 1 = -\dfrac{2}{5}(x - 6)$ or $y = -\dfrac{2}{5}x + \dfrac{17}{5}$

Sample Problems - Solutions

1. In each case, find the slope of the line determined by the two points given.

   a) $(5,-1)$ and $(1,3)$

   Solution: We will use the slope formula to find the slope. Let $(x_1,y_1)=(5,-1)$ and $(x_2,y_2)=(1,3)$. Then the slope formula gives us

   $$m=\frac{\text{rise}}{\text{run}}=\frac{y_2-y_1}{x_2-x_1}=\frac{3-(-1)}{1-5}=\frac{4}{-4}=\boxed{-1}$$

   b) $(2,1)$ and $(6,3)$

   Solution: We will use the slope formula. Let $(x_1,y_1)=(2,1)$ and $(x_2,y_2)=(6,3)$. Then the slope formula gives us

   $$m=\frac{\text{rise}}{\text{run}}=\frac{y_2-y_1}{x_2-x_1}=\frac{3-1}{6-2}=\frac{2}{4}=\boxed{\frac{1}{2}}$$

   c) $(8,-1)$ and $(-1,-1)$

   Solution: We will use the slope formula. $(x_1,y_1)=(8,-1)$ and $(x_2,y_2)=(-1,-1)$

   $$m=\frac{\text{rise}}{\text{run}}=\frac{y_2-y_1}{x_2-x_1}=\frac{-1-(-1)}{-1-8}=\frac{0}{-9}=\boxed{0}$$

   Indeed, **horizontal lines all have a slope of zero**. Since the line is horizontal, all of its points have the same $y-$coordinate and so the rise, y_2-y_1 is zero.

   d) $(7,3)$ and $(7,-4)$

   Solution: We will use the slope formula. $(x_1,y_1)=(7,3)$ and $(x_2,y_2)=(7,-4)$.

   $$m=\frac{\text{rise}}{\text{run}}=\frac{y_2-y_1}{x_2-x_1}=\frac{-4-3}{7-7}=\frac{-7}{0}=\text{undefined}$$

   Vertical lines do not have slopes. Since the line is vertical, all of its points have the same $x-$coordinate and so the run, x_2-x_1 is zero. This line has $\boxed{\text{no slope}}$.

2. For each straight line given, find its slope.

   a) the line $y=-2x+5$.

   Solution: The equation of the line is given in slope-intercept form. We simply read the coefficient of x. The slope is $\boxed{-2}$.

   b) the line $2x-5y=8$

   Solution: We first bring the equation of the line to its slope-intercept form by solving for y.. Then the coefficient of x is the slope.

   $$\begin{aligned} 2x-5y &= 8 && \text{add } 5y \\ 2x &= 5y+8 && \text{subtract } 8 \\ 2x-8 &= 5y && \text{divide by } 5 \\ \frac{2x-8}{5} &= y && \\ y &= \frac{2}{5}x-\frac{8}{5} && \text{The slope is } \boxed{\frac{2}{5}}. \end{aligned}$$

c) the line $y = 3$

Solution: The equation of the line is given in slope-intercept form. We simply read the coefficient of x. In this case, we don't see x because its coefficient is zero. We can re-write it as $y = 0x + 3$. If we are not sure, we can graph the line and see from its graph that it is horizontal. Therefore, the slope is $\boxed{3}$.

d) the line $7(x-3) + x - 2y = 2y + 1$

Solution: We first bring the equation of the line to its slope-intercept form by solving for y.. Then the coefficient of x is the slope.

$$\begin{aligned}
7(x-3)+x-2y &= 2y+1 && \text{distribute}\\
7x-21+x-2y &= 2y+1 && \text{combine like terms}\\
8x-2y-21 &= 2y+1 && \text{add } 2y\\
8x-21 &= 4y+1 && \text{subtract } 1\\
8x-22 &= 4y && \text{divide by } 4\\
\frac{8x-22}{4} &= y\\
y &= \frac{8}{4}x-\frac{22}{4} = 2x-\frac{11}{2} && \text{Thus the slope is } \boxed{2}.
\end{aligned}$$

e) the line $x = -\frac{1}{2}$.

Solution: We can not bring this equation to its slope-intercept form by solving for y because y does not appear in it. This means that there is no slope. If this is confusing, select any two points from the line, say $\left(-\frac{1}{2}, 1\right)$ and $\left(-\frac{1}{2}, 5\right)$ and apply the slope formula. It will be clear that m is undefined since we will end up dividing by zero. This line has $\boxed{\text{no slope}}$.

f) a line parallel to $y = 3x - 1$.

Solution: The line $y = 3x - 1$ has slope 3. Parallel lines have the same slope. So our line also has a slope $\boxed{3}$.

g) a line perpendicular to $y = 3x - 1$.

Solution: The line $y = 3x - 1$ has slope 3. Perpendicular lines have slopes that are negative reciprocals of each other. The negative reciprocal of 3 is $\boxed{-\frac{1}{3}}$.

3. Find an equation for the straight line that has slope 4 and $y-$intercept $(0, -5)$.

Solution: The equation is very easy to find with the data given. The slope-intercept form of the line is $y = mx + b$ and we know that $m = 4$ and $b = -5$. Thus the answer is $\boxed{y = 4x - 5}$.

4. Find an equation for the straight line that has slope $-\frac{1}{2}$ and passes through the point $(4, 1)$.

Solution: Since the point given is not the $y-$intercept, we will use the point-slope form. We know that the slope is $m = -\frac{1}{2}$ and a point on the line is $(4, 1)$. We can write the line's equation in one easy step: $\boxed{y - 1 = -\frac{1}{2}(x - 4)}$.

5. Find an equation of the straight line that is parallel to $3x-2y=12$ and passes through the point $(8,-1)$.

Solution: We start with the slope of $3x-2y=12$. We bring the equation to its slope-intercept form by solving for y.

$$\begin{aligned} 3x-2y &= 12 && \text{add } 2y \\ 3x &= 2y+12 && \text{subtract } 12 \\ 3x-12 &= 2y && \text{divide by } 2 \\ y &= \frac{3}{2}x-6 \end{aligned}$$

Thus the line $3x-2y=12$ has slope $\frac{3}{2}$. Since parallel, our line must have the same slope. Now the problem is like the previous one: the slope and a point is given. Using the point-slope form, we write

$$y-(-1)=\frac{3}{2}(x-8) \quad \text{and simplify:} \quad \boxed{y+1=\frac{3}{2}(x-8)}$$

If we are to present our answer in the slope-intercept form, that takes just a few additional steps from here.

$$\begin{aligned} y+1 &= \frac{3}{2}(x-8) && \text{distribute } \frac{3}{2} \\ y+1 &= \frac{3}{2}x-\frac{3}{2}\cdot 8 && \text{subtract } 1 \\ y &= \frac{3}{2}x-12-1 \end{aligned}$$

$$\boxed{y=\frac{3}{2}x-13}$$

6. Find an equation of the straight line that is perpendicular to $x+5y=-3$ and passes through the point $(-1,4)$.

Solution: We start with the slope of $x+5y=-3$. We bring the equation $x+5y=-3$ to its slope-intercept form by solving for y.

$$\begin{aligned} x+5y &= -3 && \text{subtract } x \\ 5y &= -x-3 && \text{divide by } 5 \\ y &= -\frac{1}{5}x-\frac{3}{5} \end{aligned}$$

Thus the line $x+5y=-3$ has slope $-\frac{1}{5}$. Since they are perpendicular, the slope of our line is the negative reciprocal of $-\frac{1}{5}$, which is 5. Now we have the slope the point given. Using the point-slope form, we write

$$y-4 = 5(x-(-1)) \quad \text{and simplify:} \quad \boxed{y-4=5(x+1)}$$

$$\begin{aligned} y-4 &= 5x+5 \\ y &= 5x+9 \end{aligned}$$

The slope-intercept form can be easily found from the point-slope form:

$$y-4=5(x+1)$$
$$y-4=5x+5$$
$$\boxed{y=5x+9}$$

7. Find an equation of the straight line that passes through the points $(3,-1)$ and $(1,5)$.

   Solution: We will presesnt two different methods.

   Method 1. Point-slope form.

   We first find the slope determined by the points. If the point $(x_1, y_1) = (3,-1)$ and $(x_{2,}y_2) = (1,5)$, then the slope formula gives us

$$m = \frac{\text{rise}}{\text{run}} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{5-(-1)}{1-3} = \frac{6}{-2} = -3$$

   We know that the slope is $m = -3$ and a point on the line is $(1,5)$. We can write the line's equation in the point-slope form in one easy step: $\boxed{y - 5 = -3(x-1)}$.

   Method 2. This is an application of systems of equations.

   We are looking for the slope-intercept form of the straight line, $y = mx + b$. We will know the answer once we have found the value of m and b. Both points $(1,5)$ and $(3,-1)$ are on the line, and so their coordinates are solution of the equation $y = mx + b$. We obtain two equations by stating that each point is on the line.

$$5 = m(1) + b \text{ since } (1,5) \text{ is on the line, and } -1 = m(3) + b \text{ since } (3,-1) \text{ is on the line}$$

   This gives us a system of linear equations in m and b:

$$\begin{cases} m + b = 5 \\ 3m + b = -1 \end{cases}$$

   This system can easily be solved by elimination. We multiply the first equation by -1 and the add the two equations

$$\begin{array}{rl} & -m - b = -5 \\ + & 3m + b = -1 \\ \hline & 2m \quad = -6 \\ & m = -3 \end{array}$$

   Now that we know the value of m, we find b by sustituting $m = -3$ into the first equation (any of the two can be used) and solve for b.

$$\begin{aligned} -3 + b &= 5 \\ b &= 8 \end{aligned}$$

   Thus $m = -3$ and $b = 8$ and so the equation of the line is $\boxed{y = -3x + 8}$.

8. Find the midpoint of the line segment determined by $A(7,-4)$ and $B(3,-2)$.

   Solution: The $x-$coordinate of the midpoint is the average of the $x-$coordinates of A and B.

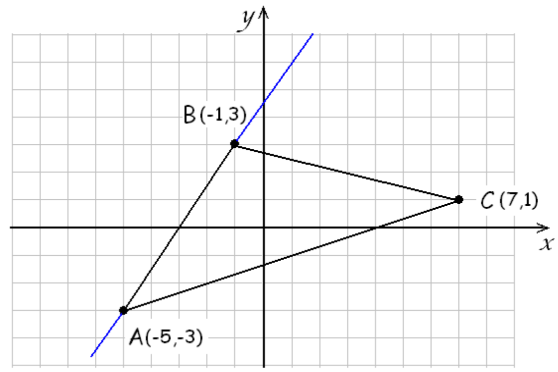
$$x_M = \frac{x_A + x_B}{2} = \frac{7+3}{2} = \frac{10}{2} = 5$$

   Similarly, the $y-$coordinate of the midpoint is the average of the $y-$coordinates of A and B.

$$y_M = \frac{y_A + y_B}{2} = \frac{-4+(-2)}{2} = \frac{-6}{2} = -3 \qquad \text{Thus the midpoint is } \boxed{M(5,-3)}.$$

9. Consider the triangle determined by $A(-5,-3)$, $B(-1,3)$, and $C(7,1)$.

a) Find an equation for the line that connects A and B.



Solution: We find the slope of line AB using the slope formula.

$$m_{AB} = \frac{y_B - y_A}{x_B - x_A} = \frac{3-(-3)}{-1-(-5)} = \frac{3+3}{-1+5} = \frac{6}{4} = \frac{3}{2}$$

We can now use the slope and point A to write the point-slope form of the equation: $\boxed{y+3 = \frac{3}{2}(x+5)}$,

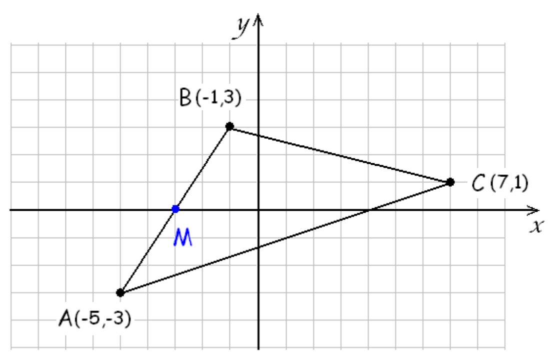
or the slope with point B: $\boxed{y-3 = \frac{3}{2}(x+1)}$.

Both solutions are correct. We can bring the equation to the slope-intercept form:

$$\begin{aligned} y+3 &= \frac{3}{2}(x+5) \\ y+3 &= \frac{3}{2}x + \frac{15}{2} \\ y &= \frac{3}{2}x + \frac{15}{2} - 3 = \frac{3}{2}x + \frac{15}{2} - \frac{6}{2} = \frac{3}{2}x + \frac{9}{2} = \frac{3}{2}x + 4\frac{1}{2} \end{aligned}$$

We can see on the picture that this equation is correct.

b) Find the midpoint of line segment AB.



Solution: The $x-$coordinate of the midpoint is the average of the $x-$coordinates of A and B.

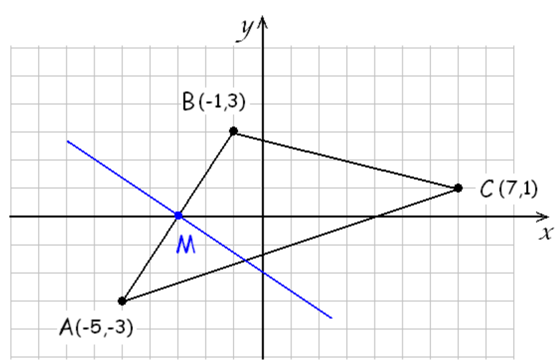
$$x_M = \frac{x_A + x_B}{2} = \frac{-5+1}{2} = \frac{-4}{2} = -2$$

Similarly, the $y-$coordinate of the midpoint is the average of the $y-$coordinates of A and B.

$$y_M = \frac{y_A + y_B}{2} = \frac{-3+3}{2} = \frac{0}{2} = 0$$

Thus the midpoint is $\boxed{M(-2,0)}$

c) Find an equation for the perpendicular bisector of side AB.



Solution: In part a), we found the slope of line AB to be $\frac{3}{2}$. The perpendicular bisector is perpendicular (duh) to this line segment and so its slope must be the negative reciprocal of $\frac{3}{2}$, which is $-\frac{2}{3}$. The perpendicular bisector contains the midpoint M of line segment AB. We found the midpoint in part b) to be $M(-2,0)$.

We can write the point-slope form of the equation: $y-(-2) = -\frac{2}{3}(x-0)$ and simplify this to

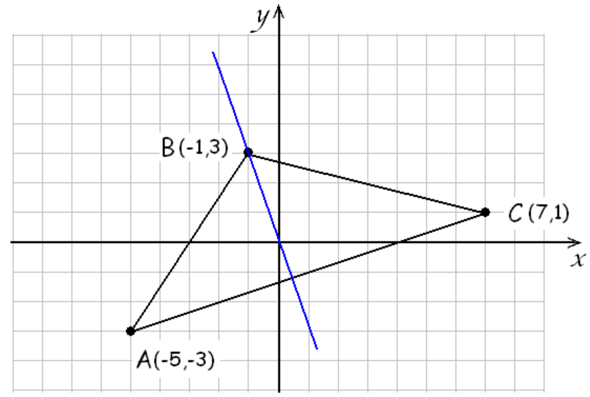
$y+2 = -\frac{2}{3}x$ or $\boxed{y = -\frac{2}{3}x - 2}$. We can see from the picture that our equation is correct.

d) Find an equation for the height (or altitude) belonging to side AC.

Solution: The height belonging to side AC is a line that is perpendicular to AC and passes through the opposite vertex, which is point B.

We compute the slope of line segment AC.

$$m_{AC} = \frac{y_C - y_A}{x_C - x_A} = \frac{1-(-3)}{7-(-5)} = \frac{1+3}{7+5} = \frac{4}{12} = \frac{1}{3}$$



The slope of the height, since perpendicular, is the negative reciprocal of $\frac{1}{3}$, which is -3. We can now write the point-slope form with slope -3 and point $B(-1,3)$. The answer is $\boxed{y-3=-3(x+1)}$.

If we are to present the slope-intecept form of the equation, we can get it with just a few steps from the slope-intercept form.

$$\begin{aligned} y-3 &= -3(x+1) \\ y-3 &= -3x-3 \\ y &= -3x \end{aligned}$$

Thus the slope-intecept form of this line is $\boxed{y=-3x}$. We can see from the picture that our equation is correct.

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