Part 1 - The History Thus Far

Recall what we know about exponentiation thus far. Exponential notation was first defined to express repeated multiplication.

Definition: We define 2^7 to denote the factor 2 multiplied by itself repeatedly, such as

$$\underbrace{2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2}_{7 \text{ factors}} = 2^7$$

When mathematicians agreed to this definition, that was a free choice. Once this definition was accepted, certain properties are automatically true, and we had no other option but to recognize them as true. They just fell into our laps, or, as we say, they are consequence of the definition.

Theorem 1. If a is any number and m, n are any positive integers, then $a^n \cdot a^m = a^{n+m}$

Theorem 2. If a is any non-zero number and m, n are any positive integers, then $\dfrac{a^n}{a^m} = a^{n-m}$

Theorem 3. If a is any number and m, n are any positive integers, then $(a^n)^m = a^{nm}$

Theorem 4. If a, b are any numbers and n is any positive integer, then $(ab)^n = a^n b^n$

Theorem 5. If a, b are any numbers, $b \neq 0$, and n is any positive integer, then $\left(\dfrac{a}{b}\right)^n = \dfrac{a^n}{b^n}$

Consider the expression 2^x. The problem was that the definition of exponentiation only allows for a positive integer value of x. The expression 2^x was meaningful for $x = 2$ or 9 or 100, but not for values of x such as -3 or $\dfrac{3}{5}$ or 3.2. In short, the world of exponents was just the set of all natural numbers. Mathematicians usually don't like that. The best case scenario, the ultimate hope is that the definition of exponents could be extended to any number for x. That way, 2^x would be meaningful, no matter what the value of x is.

In the past, we had looked at how mathematicians extended exponential notation from the set of natural numbers ($\mathbb{N}$) to the set of all integers, ($\mathbb{Z}$). They defined zero- and negative exponent in such a way so that the rules listed above still worked. Recall that this principle is called the expansion principle.

Definition: Mathematicians often try to enlarge our world, in other words, to generalize definitions and rules to a larger set. The **expansion principle** is that when we enlarge our mathematics, we do so in such a way that the new definitions never create conflicts with the mathematics we already have.

As it turned out, the demand to preserve the rule of exponents allowed only one possible definition when we stepped out from $\mathbb{N}$ to $\mathbb{Z}$.

As we stepped out to integer exponents, we insisted on a definition that does not conflict with Rule 2, $\dfrac{a^n}{a^m} = a^{n-m}$. Then there was only one possible choice for 2^0.

$$2^0 = 2^{3-3} \overset{\text{rule 2}}{=} \frac{2^3}{2^3} = \frac{8}{8} = 1$$

The same idea gave us just one possible interpretation for 2^{-3}.

$$2^{-3} = 2^{1-4} \overset{\text{Rule 2}}{=} \frac{2^1}{2^4} = \frac{2}{16} = \frac{1}{8} = \frac{1}{2^3} \quad \text{or, more elegantly, } 2^{-3} = 2^{1-4} \overset{\text{Rule 2}}{=} \frac{2^1}{2^4} = \frac{\not{2}}{\not{2} \cdot 2 \cdot 2 \cdot 2} = \frac{1}{2^3}$$

So, here we are at this point: exponentiation is defined for all integer exponents, and

Theorem 1. If a is any number and m, n are any positive integers, then $a^n \cdot a^m = a^{n+m}$

Theorem 2. If a is any non-zero number and m, n are any positive integers, then $\dfrac{a^n}{a^m} = a^{n-m}$

Theorem 3. If a is any number and m, n are any positive integers, then $(a^n)^m = a^{nm}$

Theorem 4. If a, b are any numbers and n is any positive integer, then $(ab)^n = a^n b^n$

Theorem 5. If a, b are any numbers, $b \neq 0$, and n is any positive integer, then $\left(\dfrac{a}{b}\right)^n = \dfrac{a^n}{b^n}$

Theorem 6. If a is any non-zero number, then $a^0 = 1$.

0^0 is undefined.

Theorem 7. If a is any non-zero number, and n is any positive integer, then $a^{-n} = \dfrac{1}{a^n}$.

0^{-n} is undefined.

Next, mathematicians hoped to extend exponential notation for all rational numbers, and then perhaps for all real numbers. That way, the expression 2^x would be meaningful for every real number x.

Part 2 - Rational Exponents

Today we are going to see how exponential notation can be extended to rational numbers. The driving principle here was also the expansion principle, but this time with a focus on rule 3, $(a^n)^m = a^{nm}$. We also had to make some compromises in order to keep mathematical properties that are important for us.

Consider $5^{1/3}$. At this point, this expression has no meaning, as the definition, repeated multiplication can not be applied. We have the freedom to define $5^{1/3}$ any way we like. Let us denote $5^{1/3}$ by x for now.

$$5^{\frac{1}{3}} = x$$

Considering the rule $(a^n)^m = a^{nm}$, there is a good reason for us to want to raise $5^{1/3}$ to the third power.

$$\left(5^{\frac{1}{3}}\right)^3 = (5)^{\frac{1}{3} \cdot 3} = 5^1 = 5$$

This means that if we want to define $5^{1/3}$ in a way so that the rule $(a^n)^m = a^{nm}$ will still work, then $5^{1/3}$ must be a number that, when raised to third power, the result is 5.

$$\begin{aligned} 5^{\frac{1}{3}} &= x \qquad \text{raise to the 3rd power} \\ \left(5^{\frac{1}{3}}\right)^3 &= x^3 \\ 5 &= x^3 \end{aligned}$$

There is one number that does that, and we denote it by $\sqrt[3]{x}$. Recall the definition of $\sqrt[3]{x}$: it is the number, that, when raised to the third power, we get x.

Similarly, if we want to define $5^{1/2}$, the rule $(a^n)^m = a^{nm}$ demands that $5^{1/2}$ is a number that, when squared, the result is 5.

$$\left(5^{\frac{1}{2}}\right)^2 = (5)^{\frac{1}{2}\cdot 2} = 5^1 = 5$$

So, if we denote $5^{1/2}$ by y, we have that

$$\begin{aligned} 5^{\frac{1}{2}} &= y \\ \left(5^{\frac{1}{2}}\right)^2 &= y^2 \\ 5 &= y^2 \end{aligned}$$

There are two such numbers, $\sqrt{5}$ and $-\sqrt{5}$. When we defined $\sqrt{5}$, we decided to define the positive candidate as $\sqrt{5}$. Similarly, it was not a difficult decision to go with the positive candidate, and so

$$5^{\frac{1}{2}} = \sqrt{5}$$

Both sides square to 5. The left-hand side by the rule $(a^n)^m = a^{nm}$, and the right-hand side by the definition of $\sqrt{5}$. So, we can extend exponential notation to rational numbers in the form of $a^{1/n}$ with the rule $(a^n)^m = a^{nm}$ in mind.

Theorem 8. If a is any number and n is any positive integer, then $a^{1/n} = \sqrt[n]{a}$.

There are some issues with this definition, but we already had them when we defined roots of numbers. Recall that $\sqrt{-9}$ and $\sqrt[4]{-16}$ and $\sqrt[6]{-1}$ are all expressions that can not be defined as real numbers, because any real number, positive or negative, will result in a positive number when raised to an *even* power. Therefore, $(-9)^{1/2}$ and $(-16)^{1/4}$ and $(-1)^{1/6}$ are all undefined.

Not all rational numbers are of the form $\frac{1}{n}$. How would we define something to the power of $\frac{2}{3}$? At first sight, this does not seem to be a difficult task. We could interpret $\frac{2}{3}$ as $\frac{1}{3}\cdot 2$, and then, using the same rule $(a^n)^m = a^{nm}$, we could interpret it as two different exponentiation. Consider, for example, $8^{2/3}$.

$$8^{2/3} = 8^{(1/3)\cdot 2} = \left(8^{1/3}\right)^2 = \left(\sqrt[3]{8}\right)^2 = 2^2 = 4 \quad \text{or} \quad 8^{2/3} = 8^{2\cdot(1/3)} = \left(8^2\right)^{1/3} = \sqrt[3]{8^2} = \sqrt[3]{64} = 4$$

Based on the computation above, it appears that extending exponential notation to fractions is complete, with the definition $a^{n/m} = \begin{cases} \sqrt[m]{a^n} \\ \left(\sqrt[m]{a}\right)^n \end{cases}$. This is not the case. We are facing a new issue that will force us to make more difficult choices than before.

Caution! Exponential notation was defined in *several different ways*. In advanced calculus, there will be a completely different approach to exponential notation, resulting in a different definition. The different definitions usually agree in the results, but not always. Occasionally, the different definitions lead to different results!

Here is the problem. Consider $(-8)^{4/6}$. The two computations shown in case of $8^{2/3}$ will not work the same way.

$$(-8)^{4/6} = \begin{cases} (-8)^{4\cdot(1/6)} = \left((-8)^4\right)^{1/6} = \sqrt[6]{(-8)^4} = \sqrt[6]{4096} = 4 \\ \text{or} \\ (-8)^{(1/6)\cdot 4} = \left(\sqrt[6]{-8}\right)^4 = \text{undefined} \quad \text{because } \sqrt[6]{-8} \text{ is undefined} \end{cases}$$

This is a troubling situation. If we interpret $\frac{4}{6}$ as $4 \cdot \frac{1}{6}$, we get a different result from when interpreting $\frac{4}{6}$ as $\frac{1}{6} \cdot 4$. What has happened here, that in case of a non-reduced fraction, we lose our grip on handling negative bases. There is no perfect way out of this mess.

We could define $a^{n/m}$ in such a way that we restrict the order between exponentiation $()^n$ and taking the mth root so that we only allow one order. We could also demand that fractions be always reduced when in exponents.

In intermediate algebra, we will follow a different route. We want to keep the order between exponentiation and taking the root flexible. We also demand that fractions behave the same way, whether they are in lowest terms or not. So, we simply gave up on defining fractions as exponents with negative bases. When the base is negative, the only fraction exponent allowed is of the form $\frac{1}{n}$.

Theorem 9. Suppose that m and n are integers with $m \neq 0$ and $n \neq 1$.

If a is negative, then $a^{n/m}$ is undefined. If a is non-negative, then $a^{n/m} = \begin{cases} \sqrt[m]{a^n} \\ \left(\sqrt[m]{a}\right)^n \end{cases}$.

This is the first example we saw that expanding our world came with a price. As we extended exponential notation from the set of all integers ($\mathbb{Z}$) to the set of all rational numbers ($\mathbb{Q}$), we had to give up something. This defintion results in giving up on negative bases (with the exception of exponent $\frac{1}{n}$). Other definitions of exponentiation are based on different decisions.

Part 3 - Applications

Example 1. Evaluate each of the given expressions. Present your answer using only positive integer exponents.

a) $16^{3/4}$ b) $-25^{1/2}$ c) $(-25)^{1/2}$ d) $(-8)^{1/3}$ e) $(-27)^{2/3}$ f) $27^{-2/3}$ g) $\left(\sqrt{8}\right)^{2/3}$

Solution: a) $16^{3/4} = \left(\sqrt[4]{16}\right)^3 = 2^3 = \boxed{8}$

b) Recall the difference between -2^2 and $(-2)^2$. In the first case -2^2, we are not squaring -2. The base is 2. We can interpret -2^2 as $-1 \cdot 2^2$. So, in the case of $-25^{1/2}$, the base is 25.

$-25^{1/2} = -1 \cdot 25^{1/2} = -1 \cdot \sqrt{25} = \boxed{-5}$.

c) $(-25)^{1/2} = \sqrt{-25} = \boxed{\text{undefined}}$ because no real number has a negative square.

d) $(-8)^{1/3} = \sqrt[3]{-8} = \boxed{-2}$

e) $(-27)^{2/3} = \boxed{\text{undefined}}$ because when the base is negative, the only fraction allowed is of the form $\dfrac{1}{n}$.

f) $27^{-2/3} = \dfrac{1}{27^{2/3}} = \dfrac{1}{\left(\sqrt[3]{27}\right)^2} = \dfrac{1}{3^2} = \boxed{\dfrac{1}{9}}$

g) $\left(\sqrt{8}\right)^{2/3} = \left(\left(\sqrt{8}\right)^2\right)^{1/3} = 8^{1/3} = \boxed{2}$

or $\left(\sqrt{8}\right)^{2/3} = \left(8^{1/2}\right)^{2/3} = 8^{(1/2)\cdot(2/3)} = 8^{1/3} = \sqrt[3]{8} = 2$

Notice that we have the free choice to decide the order in which we perform the exponentiaton and taking the root. On a case by case basis, one method is preferred. For example, in the case of $27^{2/3}$, we took the 3rd root first, and then we squared. This is because the third root is nice (i.e. integer) and the other order, while also correct, would make the numbers unnecessarily large. In the last example, it was more advantageous to first exponentiate, and then take the root.

Whenever we deal with complicated exponents, we need to keep in mind that when rational exponents were defined, the goal was to preserve rules of exponents. So, we can apply all rules (1-9) to these exponents.

Example 2. Simplify each of the given expressions. Present your answer using only positive integer exponents.

a) $5^{1/3} \cdot 5^{1/2}$ b) $\dfrac{8^{7/6}}{8^{1/2}}$ c) $\left(64^{2/3}\right)^{1/2}$ d) $2^{1/2} \cdot 18^{1/2}$ e) $\dfrac{96^{3/4}}{6^{3/4}}$

Solution: a) We will apply the first rule of exponents, $a^n \cdot a^m = a^{n+m}$.

$5^{1/3} \cdot 5^{1/2} = 5^{1/3+1/2} = 5^{5/6}$ as $\dfrac{1}{3} + \dfrac{1}{2} = \dfrac{2}{6} + \dfrac{3}{6} = \dfrac{5}{6}$ (printed computations look terrible in the exponents)

b) We will apply the second rule of exponents, $\dfrac{a^n}{a^m} = a^{n-m}$.

$\dfrac{8^{7/6}}{8^{1/2}} = 8^{7/6-1/2} = 8^{2/3} = \left(\sqrt[3]{8}\right)^2 = 2^2 = 4$ since $\dfrac{7}{6} - \dfrac{1}{2} = \dfrac{7-3}{6} = \dfrac{4}{6} = \dfrac{2}{3}$

c) We will apply the third rule of exponents, $(a^n)^m = a^{nm}$.

$\left(64^{2/3}\right)^{1/2} = 64^{(2/3)\cdot(1/2)} = 64^{1/3} = \sqrt[3]{64} = 4$ since $\dfrac{2}{3} \cdot \dfrac{1}{2} = \dfrac{1}{3}$

d) We will apply the fourth rule of exponents, $(ab)^n = a^n b^n$.
$2^{1/2} \cdot 18^{1/2} = (2 \cdot 18)^{1/2} = 36^{1/2} = \sqrt{36} = 2$

e) We will apply the fifth rule of exponents, $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$.
$$\frac{96^{3/4}}{6^{3/4}} = \left(\frac{96}{6}\right)^{3/4} = \left(\frac{48}{3}\right)^{3/4} = 16^{3/4} = \left(\sqrt[4]{16}\right)^3 = 2^3 = 8$$

Example 3. Simplify each of the given expressions. Present your answer using only positive integer exponents. Assume that all variables represent positive numbers.

a) $\dfrac{x^{1/2}x^{2/3}}{x^{1/6}}$ b) $\left(\dfrac{p^{2/3}p^{5/3}}{\left(-p^{1/6}\right)^2}\right)^{-1/2}$ c) $\left(\dfrac{x^{2/3}y^{-1/6}}{x^{-1/2}y^{-1/3}}\right)^{1/2}$

Solution: a) $\dfrac{x^{1/2}x^{2/3}}{x^{1/6}} = \dfrac{x^{1/2+2/3}}{x^{1/6}} = \dfrac{x^{7/6}}{x^{1/6}} = x^{7/6-1/6} = x^1 = \boxed{x}$ as $\dfrac{1}{2} + \dfrac{2}{3} = \dfrac{3+4}{6} = \dfrac{7}{6}$

b) We simplify the numerator and denominator first.
$$\left(\frac{p^{2/3}p^{5/3}}{\left(-p^{1/6}\right)^2}\right)^{-1/2} = \left(\frac{p^{2/3+5/3}}{p^{2/6}}\right)^{-1/2} = \left(\frac{p^{7/3}}{p^{1/3}}\right)^{-1/2}$$
We now perform the division.
$$\left(\frac{p^{7/3}}{p^{1/3}}\right)^{-1/2} = \left(p^{7/3-1/3}\right)^{-1/2} = \left(p^{6/3}\right)^{-1/2} = \left(p^2\right)^{-1/2}$$
We now simplify the exponent.
$$\left(p^2\right)^{-1/2} = p^{2\cdot(-1/2)} = p^{-1} = \boxed{\frac{1}{p}}$$

b) We first apply the rule $\dfrac{a^n}{a^m} = a^{n-m}$.
$$\left(\frac{x^{2/3}y^{-1/6}}{x^{-1/2}y^{-1/3}}\right)^{1/2} = \left(x^{2/3-(-1/2)}y^{-1/6-(-1/3)}\right)^{1/2} = \left(x^{7/6}y^{1/6}\right)^{1/2}$$
because $\dfrac{2}{3} - \left(-\dfrac{1}{2}\right) = \dfrac{4+3}{6} = \dfrac{7}{6}$ and $-\dfrac{1}{6} - \left(-\dfrac{1}{3}\right) = \dfrac{-1+2}{6} = \dfrac{1}{6}$

We will further simplify the expression using the rules $(ab)^n = a^n b^n$ and $(a^n)^m = a^{nm}$.
$$\left(x^{7/6}y^{1/6}\right)^{1/2} = \left(x^{7/6}\right)^{1/2}\left(y^{1/6}\right)^{1/2} = x^{(7/6)\cdot(1/2)}y^{(1/6)\cdot(1/2)} = x^{7/12}y^{1/12}$$
We need to present this expression using only positive integer exponents. This will take a few more steps.
$$x^{7/12}y^{1/12} = \left(x^7\right)^{1/12}(y)^{1/12} = \left(x^7 y\right)^{1/12} = \boxed{\sqrt[12]{x^7 y}}$$

Rational exponents are also useful in simplifying complicated radical expressions.

Example 4. Simplify each of the given expressions. Present your answer using only positive integer exponents. Assume that x represents a positive number.

a) $\sqrt{x}\,\sqrt[3]{x}\,\sqrt[4]{x^3}$ b) $\sqrt{x\sqrt[3]{x^2\,\sqrt[4]{x^{-3}}}}$ c) $\sqrt[18]{x^{12}}$

Solution: a) We first re-write the radicals as exponents. $\sqrt{x}=x^{1/2}$ and $\sqrt[3]{x}=x^{1/3}$ and $\sqrt[4]{x^3}=x^{3/4}$.

$$\sqrt{x}\,\sqrt[3]{x}\,\sqrt[4]{x^3}=x^{1/2}x^{1/3}x^{3/4}=x^{(1/2)+(1/3)+(3/4)}=x^{19/12}=\boxed{\sqrt[12]{x^{19}}}\quad\text{as}\quad\frac{1}{2}+\frac{1}{3}+\frac{3}{4}=\frac{6+4+9}{12}=\frac{19}{12}$$

b) We re-write the radicals as exponents.

$$\sqrt{x\sqrt[3]{x^2\,\sqrt[4]{x^{-3}}}}=\left(x\left(x^2\,x^{-3/4}\right)^{1/3}\right)^{1/2}\qquad 2-\frac{3}{4}=\frac{8-3}{4}=\frac{5}{4}$$

$$=\left(x\left(x^{5/4}\right)^{1/3}\right)^{1/2}=\left(x\cdot x^{(5/4)\cdot(1/3)}\right)^{1/2}\qquad \frac{5}{4}\cdot\frac{1}{3}=\frac{5}{12}\text{ and }1+\frac{5}{12}=\frac{17}{12}$$

$$=\left(x\cdot x^{5/12}\right)^{1/2}=\left(x^{17/12}\right)^{1/2}=x^{(17/12)\cdot(1/2)}=x^{17/24}=\boxed{\sqrt[24]{x^{17}}}$$

c) $\sqrt[18]{x^{12}}=x^{12/18}=x^{2/3}=\boxed{\sqrt[3]{x^2}}$

Practice Problems

Simplify each of the following expressions.

1. $9^{1/2}$
2. $9^{-1/2}$
3. $-9^{1/2}$
4. $-9^{-1/2}$
5. $(-9)^{1/2}$
6. $9^{3/2}$
7. $8^{1/3}$
8. $8^{-1/3}$
9. $(-8)^{1/3}$
10. $16^{-3/4}$
11. $5^{1/2}$
12. $-4^{-5/2}$
13. $(-32)^{1/5}$
14. $32^{2/5}$
15. $36^{1/2}$
16. $36^{-1/2}$
17. $-36^{1/2}$
18. $(-36)^{1/2}$
19. $(-36)^{-1/2}$
20. $36^{-3/2}$
21. $-4^{-3/2}$
22. $(-4)^{-3/2}$
23. $(-8)^{1/3}$
24. $(-8)^{2/3}$
25. $-16^{1/2}$
26. $(-16)^{1/2}$
27. $8^{-2/3}$
28. $(32)^{-4/5}$
29. $(-32)^{-4/5}$
30. $81^{-3/4}$

Simplify each of the following. Present your final answers using positive exponents only. For example, $x^{-2/3} = \dfrac{1}{\sqrt[3]{x^2}}$. Assume that all variables represent positive numbers.

31. $\left(x^{1/3}\right)^6$
32. $\dfrac{x^{2/5}x^{-3/4}}{x^{-1/2}}$
33. $\left(a^{-1/2}\right)^{-4/5}$
34. $\dfrac{\left(m^{1/2}\right)\left(m^{3/2}\right)}{m^{-3}}$
35. $\sqrt[3]{\dfrac{x^2y^{-1}}{\left(-x^4y^{-2}\right)^2}}$
36. $\left(-8a^6b^{-24}\right)^{1/3}$
37. $\left(\dfrac{16x^4y^{\frac{2}{3}}}{36x^2y^2}\right)^{\frac{3}{2}}$
38. $\left(-3x^{-1/2}y^{3/8}x^{-7/2}\right)^0$
39. $\left(A^{2/3}\right)^{3/2}$
40. $\dfrac{\left(y^{-3/4}\right)^{-8}}{\left(y^{-1/2}\right)^{10}}$

Use exponential notation to simplify each of the following. Present your answer using positive integers only.

41. $\sqrt[8]{x^6}$
42. $\left(\sqrt[6]{y}\right)^9$
43. $\left(\sqrt[6]{2}\right)^{18}$
44. $\left(\sqrt{x}\right)\left(\sqrt[3]{x}\right)$
45. $\dfrac{\sqrt[5]{x}\,\sqrt[3]{x^2}}{x\,\sqrt[6]{x}}$
46. $\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{x}}}}}$

Answers

1. 3 2. $\dfrac{1}{3}$ 3. -3 4. $-\dfrac{1}{3}$ 5. undefined 6. 27 7. 2 8. $\dfrac{1}{2}$ 9. -2 10. $\dfrac{1}{8}$ 11. $\sqrt{5}$

12. $-\dfrac{1}{32}$ 13. -2 14. 4 15. 6 16. $\dfrac{1}{6}$ 17. -6 18. undefined 19. undefined

20. $\dfrac{1}{216}$ 21. $-\dfrac{1}{8}$ 22. undefined 23. -2 24. undefined 25. -4 26. undefined 27. $\dfrac{1}{4}$

28. $\dfrac{1}{16}$ 29. undefined 30. $\dfrac{1}{27}$ 31. x^2 32. $\sqrt[20]{x^3}$ 33. $\sqrt[5]{a^2}$ 34. m^5 35. $\dfrac{y}{x^2}$ 36. $\dfrac{-2a^2}{b^8}$

37. $\dfrac{8x^3}{27y^2}$ 38. 1 39. A 40. y^{11} 41. $\sqrt[4]{x^3}$ 42. $\left(\sqrt{y}\right)^3$ 43. 8 44. $\sqrt[6]{x^5}$

45. $\dfrac{\sqrt[5]{x}\,\sqrt[3]{x^2}}{x\,\sqrt[6]{x}} = x^{\left(\frac{1}{5}+\frac{2}{3}-\left(1+\frac{1}{6}\right)\right)} = x^{-\frac{3}{10}} = \dfrac{1}{\sqrt[10]{x^3}}$

46. $\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{x}}}}} = \left(\left(\left(\left(x^{\frac{1}{2}}\right)^{\frac{1}{2}}\right)^{\frac{1}{2}}\right)^{\frac{1}{2}}\right)^{\frac{1}{2}} = x^{\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}} = x^{\frac{1}{32}} = \sqrt[32]{x}$

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