Let us first review the usual steps taken to solve an application problem.

Step 1. Read the question carefully. Choose a quantity to be labeled as x.

Step 2. Read the question. Label all other quantities in terms of x.

Step 3. Based on the text of the problem, set up an equation in x.

Step 4. The word problem is over for a while. Solve the equation for x.

Step 5. Given the values of x, find the values of all quantities labeled in terms of x. Interpret your results within the context of the problem.

Step 6. Read the question again. State the answer.

Step 7. Check your result against the conditions stated in the problem.

If the equation we set up is quadratic, (i.e. of degree two), we could end up with two solutions, one solution, or no solutions. It is interesting to note that sometimes only one of the two solutions work, sometimes both, or neither. This makes interpreting the results more interesting and more important, as we will see in some of the examples.

The following example illustrates the difference between an application of linear and quadratic equations. We will usually end up with quadratic equations when we need to multily two unknown quantity, both labeled in terms of x.

Example 1. a) The sum of two numbers is 38, their difference is 14. Find these numbers.

b) The product of two numbers is 640, their difference is 12. Find these numbers.

Solution: When we decide to label an unknown quantity by x, we should not expect to always succeed. Sometimes the key issue is what exactly to call x. We need to use trial and error until we find a labeling that leads to the solution. It is also important that you **state in writing** what you called x.

Let us denote the smaller number by x. Then the larger number is $x + 14$, since the difference between the two numbers is 14. The equation then is

$$\underbrace{x}_{\text{smaller number}} + \underbrace{x+14}_{\text{larger number}} = 38 \qquad \text{solve for } x$$

$$\begin{aligned} 2x + 14 &= 38 && \text{subtract } 14 \\ 2x &= 24 && \text{divide by } 2 \\ x &= 12 \end{aligned}$$

Thus the smaller number, labeled x is 12. The larger number was labeled $x + 14$, so it must be $12 + 14 = 26$. Thus the numbers are 12 and 26.

We check: the difference between 26 and 12 is $26 - 12 = 14$, and their sum is indeed $26 + 12 = 38$. Thus our solution is correct. The numbers are $\boxed{12 \text{ and } 26}$.

Notice that we did not check $x = 12$ against the equation. This is because the equation was not given, we set it up. If we correctly solved an equation that is incorrectly set up, we would not catch that error. So we hold on, figure out the value of all quantities, and then check directly against the conditions stated in the problem.

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b) The product of two numbers is 640. Their difference is 12. Find these numbers.

Solution: Let us label the smaller number as x. Then the larger number is $x + 12$. The equation is

$$\begin{aligned} x(x+12) &= 640 \qquad \text{Solve for } x \\ x^2 + 12x &= 640 \\ x^2 + 12x - 640 &= 0 \end{aligned}$$

We will solve this quadratic equation by completing the square. There are numerous other methods. Half of the linear coefficient is 6.

$$\begin{aligned} x^2 + 12x - 640 &= 0 \qquad (x+6)^2 = x^2 + 12x + 36 \\ \underbrace{x^2 + 12x + 36} - 36 - 640 &= 0 \\ (x+6)^2 - 676 &= 0 \qquad \sqrt{676} = 26 \\ (x+6)^2 - 26^2 &= 0 \\ (x+6+26)(x+6-26) &= 0 \\ (x+32)(x-20) &= 0 \quad \Longrightarrow \quad x_1 = -32 \text{ and } x_2 = 20 \end{aligned}$$

f $x = -32$, then the larger number is $-32 + 12 = -20$. If $x = 20$, then the larger number is $20 + 12 = 32$. **The two solutions of the equation do not determine a pair of numbers: they are the smaller numbers in two possible pairs!** The answer is: $\boxed{20 \text{ with } 32 \text{ and } -32 \text{ with } -20}$. We check both solutions.

Checking the pair $20, 32$
$20 \cdot 32 = 640$ and $32 - 20 = 12$ ✓

Checking the pair $-32, -20$
$-20(-32) = 640$ and $-20 - (-32) = 12$ ✓

In the previous exaple, it was easy to see why if two numbers like 20 and 32 work, why then their opposites will work too. But more often than not, the two solutions of a quadratic application have a more complex relationship.

Example 2. Six times a number is equal to twice the square of the same number. Find this number.

Solution: Let us denote the number by x. Then the equation expresses the relationship stated in the problem.

$$6x = 2x^2$$

Since this is a quadratic equation, we reduce one side to zero, factor the other side and apply the zero product rule.

$$\begin{aligned} 0 &= 2x^2 - 6x \\ 2x^2 - 6x &= 0 \qquad \text{factor out the GCF, } 2x \\ 2x(x-3) &= 0 \end{aligned}$$

$x_1 = 0$ and $x_2 = 3$. We check. If $x = 0$, then twice the square of zero is zero, and so is six times zero. So, $x = 0$ works. If $x = 3$, then six times the number is 18. That is also twice the square of 3. So, both $\boxed{0 \text{ and } 3}$ work.

Example 3. Four times a number is three greater than the square of the number. Find this number.

Solution: Let us denote this number by x. Then four times the number is $4x$. The square of the number is x^2. The equation will state the relationship between those two quantities.

$$\begin{aligned}
4x &= x^2+3 \\
0 &= x^2-4x+3 \\
x^2-4x+3 &= 0 \qquad (x-2)^2 = x^2-4x+4 \\
\underbrace{x^2-4x+4}-4+3 &= 0 \\
(x-2)^2-1 &= 0 \\
(x-2+1)(x-2-1) &= 0 \\
(x-1)(x-3) &= 0 \quad \Longrightarrow \quad x_1=1 \text{ and } x_2=3
\end{aligned}$$

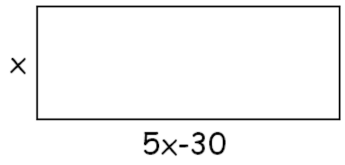
We check: if $x=1$, then $1^2=1$ is indeed three less than four times the number, $4\cdot 1=4$. And if $x=3$, then four times the number is 12 - that is indeed 3 greater than the square of 3. So, both $\boxed{1 \text{ and } 3}$ work.

The following example illustrates the kind of interesting complications we will see when interpreting the solution(s) of the equation.

Example 5. One side of a rectangle is $30\,\text{ft}$ shorter than five times another side. Find the sides if the area is $1080\,\text{ft}^2$.

Solution: Let us denote the shorter side by x. Then the longer side is $5x-30$. The equation expresses the area. Since this is a quadratic equation, our only method of solving it is by the zero product rule. There are several factoring techniques available. We will factor by completing the square.

$$\begin{aligned}
x(5x-30) &= 1080 \qquad \text{solve for } x \\
5x^2-30x &= 1080 \\
5x^2-30x-1080 &= 0 \qquad \text{factor out } 5 \\
5\left(x^2-6x-216\right) &= 0 \qquad (x-3)^2 = x^2-6x+9 \\
5\left(\underbrace{x^2-6x+9}-9-216\right) &= 0 \\
5\left((x-3)^2-225\right) &= 0 \\
5\left((x-3)^2-15^2\right) &= 0 \\
5(x-3+15)(x-3-15) &= 0 \\
5(x+12)(x-18) &= 0 \\
x_1=-12, \quad x_2 &= 18
\end{aligned}$$



$x=-12$ works with the equation. If x is -12, then the other side would be $5x-30 = 5(-12)-30 = -90$ and the area would be $(-12)(-90) = 1080$. So, $x=-12$ works algebraically, but does not work within the context of the problem, because x denotes a distance and distances can not be negative. We have to do the work of interpreting the solutions of the equation within the context of the problem.

We are left with $x=18$. Then the other side is $5(18)-30 = 60$. So, our solution is that the sides are $\boxed{18 \text{ feet and } 60 \text{ feet long}}$. We check: $18\,\text{ft}\,(60\,\text{ft}) = 1080\,\text{ft}^2$, so the area is what it should be. Also, $5(18\,\text{ft})-30\,\text{ft} = 60\,\text{ft}$, and so our solution is correct.

The following is an example with a real-world formula.

Example 6. We throw an object upward from the top of a 1200 ft tall building. The height of the object, (measured in feet) t seconds after we threw it is

$$h = -16t^2 + 160t + 1200$$

a) Where is the object 3 seconds after we threw it?

b) How long does it take for the object to hit the ground?

Solution: a) Where is the object 3 seconds after we threw it?

We need to compute h when $t = 3$. This means that we substitute 3 into t and evaluate the algebraic expression.

$$\begin{aligned} h &= -16 \cdot 3^2 + 160 \cdot 3 + 1200 = -16 \cdot 9 + 160 \cdot 3 + 1200 \\ &= -144 + 480 + 1200 = 336 + 1200 = 1536 \end{aligned}$$

Thus the object is at a height of 1536 ft after 3 seconds.

b) How long does it take for the object to hit the ground?

We need to solve the equation $t = ?$ so that $h(t) = 0$

$$\begin{aligned} h(t) &= 0 \\ -16t^2 + 160t + 1200 &= 0 \quad \text{factor out } -16 \\ -16\left(t^2 - 10t - 75\right) &= 0 \end{aligned}$$

We will factor $t^2 - 10t + 75$ by completing the square.

$$\begin{aligned} -16\left(t^2 - 10t - 75\right) &= 0 && (t-5)^2 = t^2 - 10t + 25 \quad \text{smuggle in } 25 \\ -16\left(\underbrace{t^2 - 10t + 25}\, -25 - 75\right) &= 0 && \\ -16\left((t-5)^2 - 100\right) &= 0 && \text{re-write } 100 \text{ as } 10^2 \\ -16\left((t-5)^2 - 10^2\right) &= 0 && \text{factor via the difference of squares theorem} \\ -16\,(t-5+10)\,(t-5-10) &= 0 && \text{simplify} \\ -16\,(t+5)\,(t-15) &= 0 && \text{apply zero property} \end{aligned}$$

$$t = -5 \quad \text{or} \quad t = 15$$

Since the negative solution, $t = -5$ does not make sense in the context of the problem, it is ruled out. We check $t = 15$:

$$\begin{aligned} h &= -16 \cdot 15^2 + 160 \cdot 15 + 1200 = -16 \cdot 225 + 160 \cdot 15 + 1200 \\ &= -3600 + 2400 + 1200 = -1200 + 1200 = 0 \end{aligned}$$

Thus the answer is: it will take $\boxed{15 \text{ seconds}}$ for the object to hit the ground.

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Practice Problems

1. The product of two numbers is 65. Their difference is 8. Find these numbers.
2. The square of a number is 21 more than four times the number.
3. If we square a number, we get six times the number. Find all numbers with this property.
4. If we raise a number to the third power, we get four times the number. Find all numbers with this property.
5. The product of two consecutive even integers is 840. Find these numbers.
6. The area of a rectangle is $1260\,\mathrm{m}^2$. Find the dimensions of the rectangle if we know that one side is $48\,\mathrm{m}$ longer than three times the other side.
7. We are standing on the top of a $1680\,\mathrm{ft}$ tall building and throw a small object upwards. At every second, we measure the distance of the object from the ground. Exactly t seconds after we threw the object, its height, (measured in feet) is

$$h = -16t^2 + 256t + 1680$$

   a) Find h when $t = 3$.

   b) How much does the object travel during the two seconds between 5 seconds and 7 seconds?

   c) How long does it take for the object to reach a height of $2640\,\mathrm{ft}$?

   d) How long does it take for the object to hit the ground?

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Answers

Practice Problems

1. 5 with 13 and -13 with -5 2. $-3, 7$ 3. $0, 6$ 4. $-2, 0, 2$ 5. $28, 30$ and $-30, -28$

6. $14\,\text{m}$ by $90\,\text{m}$ 7. a) $2304\,\text{ft}$ b) $128\,\text{ft}$ c) 6 seconds and 10 seconds d) 21 seconds

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