The Axioms of Real Numbers

A **definition** is a type of statement in which we agree how we will refer to things. It is true in a sense because it just sets an agreement about labeling things.

An **axiom** is a statement that we accept as true, without requiring proof of it. It usually agrees with our natural instincts and they "feel true". One example is the statement: "Two points uniquely determine a straight line".

A **theorem** is a statement that we prove to be true. But what does it mean to prove something? It means to derive it from the axioms. Mathematicians set down a set of basic 'truths', the axioms. Everything we prove, we derive them from the axioms. All theorems about the real numbers can be derived from the following axioms.

A1. Addition is commutative.
For all $x, y \in \mathbb{R}$, $x + y = y + x$.

A2. Addition is associative.
For all $x, y, z \in \mathbb{R}$, $(x + y) + z = x + (y + z)$.

A3. Additive identity.
There exists a real number d, such that for every real number x, $x + d = x$. (This number is 0).

A4. Additive inverse.
For all $x \in \mathbb{R}$, there exists a real number x^*, such that $x + x^* = 0$. (We denote this number by $-x$).

M1. Multiplication is commutative.
For all $x, y \in \mathbb{R}$, $xy = yx$.

M2. Multiplication is associative.
For all $x, y, z \in \mathbb{R}$, $(xy)\,z = x\,(yz)$.

M3. Multiplicative identity.
There exists a real number d, such that for all $x \in \mathbb{R}$, $xd = x$. (This number is 1.)

M4. Multiplicative inverse.
For all $x \in \mathbb{R}$, $x \neq 0$, there exists a real number x^*, such that $xx^* = 1$. (We denote this number by $\frac{1}{x}$).

D1. Distributive Law.
For all $x, y, z \in \mathbb{R}$, $z\,(x + y) = zx + zy$.

Sets that have these properties are called fields, and are studied in depth in abstract algebra.

The real number system also has the following axioms about ordering.

O1. For all $x, y \in \mathbb{R}$, either $a \leq b$ or $b \leq a$ or both.

O2. If $a \leq b$ and $b \leq a$, then $a = b$.

O3. If $a \leq b$ and $b \leq c$, then $a \leq c$.

O4. If $a \leq b$ then $a + c \leq b + c$.

O5. If $a \leq b$ and $0 \leq c$, then $ac \leq bc$.

A set with all these properties is called an ordered field. However, so far these properties hold for both $\mathbb{Q}$ (the set of all rational numbers) and $\mathbb{R}$ (the set of all real numbers.) What distinguishes these two is the completeness property, that is true for $\mathbb{R}$ but not for $\mathbb{Q}$.

Completeness property: Every non-empty subset of $\mathbb{R}$ that is bounded above has a least upper bound.

Identity and inverse elements

An identity element does 'nothing'. It is a unique elemet of the set that works for every element.
The inverse of an element is another element that when the operation applied, results in the identity. The inverse is different for different numbers.

		Real Numbers with Addition	Real Numbers with Multiplication
	Notation	$\langle \mathbb{R}, + \rangle$	$\langle \mathbb{R}, \cdot \rangle$
Identity	Systematic Name	additive identity	multiplicative identity
	Defining Property	does 'nothing' in addition	does 'nothing' in multiplication
	Value	0	1
Inverse	Systematic Name	additive inverse of a	multiplicative inverse of a
	Non-Systematic Name	opposite of a	reciprocal of a
	Defining Property	$a + (\text{opposite of } a) = 0$ i.e. 'takes' a to the identity	$a \cdot (\text{reciprocal of a}) = 1$ i.e. 'takes' a to the identity
	Notation	$-a$	$\dfrac{1}{a}$

 Last Revised: June 5, 2009