

In this section, we will learn about some new set operations.

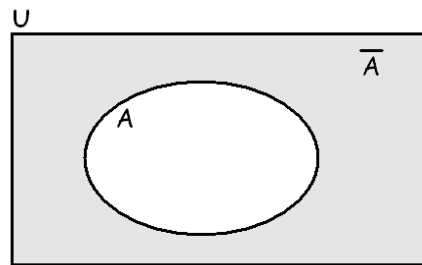
Definition: The set U usually denotes the universal set, i.e. the set everything that exists within the context of a problem.

If U is given as $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, then we will work out the problem as if only those 10 numbers exist. For that problem, this is all that exists. Until now we did not need the universal set. However, we will need it for our newly defined set operation.

Definition: Given a set A , the **complement of A** , (denoted by $\bar{A}$ or A' or A^C) is the set such that for all $x \in U$, x is an element of $\bar{A}$ if and only if x is not an element of A .

Example 1. If $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ and $A = \{1, 2, 3, 5, 8, 10\}$, compute $\bar{A}$.

Solution: If $A = \{1, 2, 3, 5, 8, 10\}$, then $\bar{A} = \{4, 6, 7, 9\}$. The picture shows what region of the Venn diagram is the complement of A .



Example 2. If $U = \mathbb{N}$ and $E = \{2k : k \in \mathbb{N}\}$, find $\bar{E}$.

Solution: $E = \{2, 4, 6, 8, 10, 12, \dots\}$ the set of all even natural numbers. To find $\bar{E}$, we need to list all natural numbers not in E . Therefore, $\bar{E} = \{1, 3, 5, 7, 9, 11, \dots\}$.

Example 3. Suppose that $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, $A = \{1, 4, 6, 10\}$, and $B = \{2, 3, 6, 7, 8, 10\}$. Find each of the following.

- a) $\overline{A \cup B}$ b) $\bar{A} \cup \bar{B}$ c) $\overline{A \cap B}$ d) $\bar{A} \cap \bar{B}$

Solution: a) We first find $A \cup B$ and then take that set's complement.

$$A \cup B = \{1, 2, 3, 4, 6, 7, 8, 10\} \text{ and then } \overline{A \cup B} = \{5, 9\}$$

b) We first find the complement of A and B and then take the union of the two sets.

$$\begin{aligned} \bar{A} &= \{2, 3, 5, 7, 8, 9\} \text{ and } \bar{B} = \{1, 4, 9\} \\ \bar{A} \cup \bar{B} &= \{2, 3, 5, 7, 8, 9\} \cup \{1, 4, 9\} = \{1, 2, 3, 4, 5, 7, 8, 9\} \end{aligned}$$

c) We first find $A \cap B$ and then take that set's complement.

$$A \cap B = \{6, 10\} \text{ and then } \overline{A \cap B} = \{1, 2, 3, 4, 5, 7, 8, 9\}$$

d) We first find the complement of A and B and then take the intersection of the two sets.

$$\begin{aligned}\overline{A} &= \{2, 3, 5, 7, 8, 9\} \text{ and } \overline{B} = \{1, 4, 5, 9\} \\ \overline{A} \cap \overline{B} &= \{2, 3, 5, 7, 8, 9\} \cap \{1, 4, 5, 9\} = \{5, 9\}\end{aligned}$$

Definition: Given a set A , the **cardinality of A** , (denoted by $|A|$ or by $n(A)$) is the number of elements in A .

Example 4. Suppose that $A = \{1, 2, 5, 7\}$. Find $n(A)$.

Solution: We simply count the elements in A . Since A has four elements, $n(A) = 4$.

Example 5. List all subsets of the given set if

a) $A = \{1\}$ b) $B = \{1, 2\}$ c) $C = \{1, 2, 3\}$ d) $D = \{1, 2, 3, 4\}$

Solution: a) The set $\{1\}$ has two subsets: $\emptyset$ and $\{1\}$.

b) The set $\{1, 2\}$ has four subsets: $\emptyset$, $\{1\}$, $\{2\}$, and $\{1, 2\}$.

c) As the set given becomes larger, it becomes important to be systematic when listing the subsets. Let us list the subsets of C by organizing by their cardinalities.

0-element subsets:	$\emptyset$	
1-element subsets:	$\{1\}, \{2\}, \{3\}$	
2-element subsets:	$\{1, 2\}, \{1, 3\}, \{2, 3\}$	
3-element subsets:	$\{1, 2, 3\}$	So there are 8 subsets.

d) We will list the subsets of D by organizing by their cardinalities.

0-element subsets:	$\emptyset$	1 subset
1-element subsets:	$\{1\}, \{2\}, \{3\}, \{4\}$	4 subsets
2-element subsets:	$\{1, 2\}$ $\{1, 3\} \quad \{2, 3\}$ $\{1, 4\} \quad \{2, 4\} \quad \{3, 4\}$	6 subsets
3-element subsets:	$\{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\}$	4 subsets
4-element subsets:	$\{1, 2, 3, 4\}$	1 subset

So there are 16 subsets.

The reader might have noticed that the number of subsets doubles when we add a new element to the set. This is indeed true.

This might sound strange at first, but we can also multiply sets.

Definition: Given non-empty sets A and B , the cartesian product of A and B , denoted by $A \times B$ is the set of all ordered pairs (a, b) where $a \in A$ and $b \in B$.

Example 6. Suppose that $P = \{1, 2, 3\}$ and $S = \{T, F\}$. Find $P \times S$ and $S \times P$.

Solution: $P \times S$ is the set of all ordered pairs (p, s) where $p \in P$ and $s \in S$. This set is:

$$P \times S = \{(1, T), (2, T), (3, T), (1, F), (2, F), (3, F)\}$$

On the other hand, the set $S \times P$ is the set of all ordered pairs (s, p) where $s \in S$ and $p \in P$. This set is:

$$S \times P = \{(T, 1), (T, 2), (T, 3), (F, 1), (F, 2), (F, 3)\}$$

Notice that this sort of multiplication is not commutative, i.e. $P \times S \neq S \times P$. We can also see why the name *product*: $n(A \times B) = n(A) \cdot n(B)$.

Part 2 - Venn Diagrams as a Problem Solving Tool

The Venn diagram is also a very useful problem solving tool.

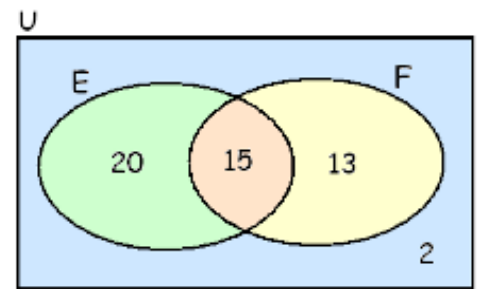
Example 7. There are 50 students in the senior class. This semester, 35 students are taking English, 28 are taking French, and 15 are taking both English and French. How many students are taking neither English, nor French?

Solution: We will approach this problem using a Venn diagram with two sets. Let E be the set of students taking English and F the set of students taking French. We draw a Venn diagram with E and F .

There are four regions on the picture. The orange-brown region will represent the students taking both English and French. The green region is for students taking *only* English; i.e. they take English but not French. Similarly, the yellow region is for students who take French but not English. The blue region is for students taking neither of these two subjects.

The orange-brown region must have 15 students in it - this is stated in the problem. For the green region, 35 is a good start, but not the correct number. The issue is that the 35 taking English includes those 15 students taking both. The orange-brown region and the green region *together* should account for the students taking English. Therefore, we must enter $35 - 15 = 20$ into the green region. Similarly, the yellow region should represent $28 - 15 = 13$ students.

If we add all three numbers, we get $20 + 15 + 13 = 48$. This accounts for all students taking at least one class. All together, we asked 50 students. This means that only 2 students are taking neither of this class.



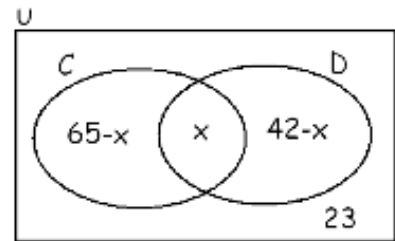
Discussion: On a Venn diagram with two sets, E and F , there are 4 regions. What would this number be in a Venn diagram with three sets, A , B , and C ? And four sets? Draw Venn diagrams where every possible region occurs exactly once.

Example 8. Consider the Venn diagram with E and F , as in the previous example. Express each of the regions using sets and set operations.

Solution: The orange-brown region is clearly $E \cap F$. The green region contains elements that are in E but not in F . That is $E \cap \overline{F}$. Similarly, the yellow region is $\overline{E} \cap F$. The blue region is $\overline{E} \cap \overline{F}$.

Example 9. There are 100 students in our Freshman class. We asked them about their pets. 23 students had no pets. 65 students had cats, and 42 students had dogs. How many students have both cat(s) and dog(s)?

Solution: We will complete the Venn diagram from starting with the intersection. Let x denote the number of students in the intersection. Then the students that have only cats must be $65 - x$. Similarly, the number of students who have only dogs is $42 - x$. The four regions together must account for all 100 students. We write an equation expressing this and then solve for x .



$$\begin{aligned} \text{both} \quad x + \text{only cats} \quad (65 - x) + \text{only dogs} \quad (42 - x) + 23 &= 100 \\ 130 - x &= 100 \\ x &= 30 \end{aligned}$$

Therefore, 30 students had both cat(s) and dog(s).



Practice Problems

- This problem shows why we need the universal set when computing the complement of a set. Suppose that $A = \{1, 3, 4\}$
 - Compute $\bar{A}$ if $U = \{1, 2, 3, 4, 5\}$.
 - Compute $\bar{A}$ if $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$.
- Let $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, $A = \{1, 2, 4, 6, 9, 10\}$, $B = \{2, 3, 4, 8, 10\}$, and $C = \{2, 7, 8, 10\}$. Find each of the following.
 - $\bar{A} \cap \bar{B} \cup \bar{C}$
 - $\bar{A} \cap (B \cup \bar{C})$
 - $(\bar{A} \cap B) \cup \bar{C}$
- List all subsets of $A = \{1, 2, 3, 4, 5\}$
- Let $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, $A = \{1, 3, 5, 7\}$, $B = \{1, 3, 4, 6, 8\}$, and $C = \{1, 4, 5, 6, 7, 9\}$. Find each of the following.

a) $A \cap B \cap C$	c) $B \cap \bar{C}$	e) $\overline{A \cap B}$	g) $(A \cap \bar{B}) \cup (C \cap \bar{B})$
b) $C \cap (B \cup A)$	d) $\bar{A} \cup \bar{B}$	f) $(B \cap \bar{C}) \cup C$	h) $n(A \cup C)$
- Let $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, $A = \{1, 3, 5, 7\}$, $B = \{1, 3, 4, 6, 8\}$, and $C = \{1, 4, 5, 6, 7, 9\}$. Label each of the following statements as true or false.

a) $9 \in \bar{A}$	d) $n(A) + n(\bar{A}) = n(U)$	g) $A \cap \bar{A} = \emptyset$	j) $\overline{A \cup B} \subseteq \bar{A} \cap \bar{B}$
b) $A \subseteq (B \cup C)$	e) $\overline{(\bar{A})} = A$	h) $\overline{A \cap B} = \bar{A} \cap \bar{B}$	k) $\bar{A} \cap \bar{B} = \overline{A \cup B}$
c) $n(A \cup B) = n(A) + n(B)$	f) $A \cup \bar{A} = U$	i) $\overline{A \cup B} = \bar{A} \cup \bar{B}$	l) $\bar{B} \cap \bar{C} = \overline{B \cup C}$
- List all two-element subsets of $C = \{1, 4, 5, 6, 7, 9\}$.
- Suppose that $A = \{1, 2, 3, 4\}$ and $B = \{p, q\}$. Find $A \times B$ and $B \times A$.

8. Let $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$, $A = \{2, 5, 6, 7, 9, 11\}$, $B = \{1, 5, 6, 8, 9, 10, 12\}$, and $C = \{2, 4, 5, 7, 9, 12\}$
- a) Draw a Venn diagram depicting these sets. d) Find $\overline{(A \cup B) \cap C}$
b) Find $(B \cup C) \cap \bar{A}$ e) True or false? $(A \cap B) \cap \bar{C} \subseteq (A \cup B) \cap \bar{C}$
c) Find $(A \cap \bar{C}) \cup (\bar{A} \cap C)$
9. We asked 50 students whether they watch TV and/or listen to radio. 38 students claimed that they watch TV, and 31 students claimed that they listen to radio. If 25 students watch TV and listen to radio, how many students do neither of these?
10. Our class of 50 is specializing in languages. 28 students are taking Spanish classes, 25 students take French classes, and 2 students are taking none of these two classes. How many students are taking both classes?
11. We asked 120 students whether they are currently taking astronomy or biology or chemistry. 46 students take astronomy, 57 take biology, and 52 take chemistry. 22 students take astronomy and biology, 18 take astronomy and chemistry, and 24 take biology and chemistry. 10 students are taking all three.
- a) Draw a Venn diagram depicting the information given.
b) How many students are taking at least two of these courses?
c) How many students are taking none of these courses?
d) How many students take biology or chemistry?
e) How many students take astronomy and biology but not chemistry?
f) How many students take exactly two of these courses?



Answers

1. a) $\{2, 5\}$ b) $\{2, 5, 6, 7, 8, 9, 10\}$

2. a) $\{5\}$ b) $\{3, 5, 8\}$ c) $\{1, 3, 4, 5, 6, 8, 9\}$

3. There are 32 subsets, listed below.

0-element subsets: $\emptyset$

1 subset

1-element subsets: $\{1\}, \{2\}, \{3\}, \{4\}, \{5\}$

5 subsets

2-element subsets:
 $\{1, 2\}$
 $\{1, 3\} \quad \{2, 3\}$
 $\{1, 4\} \quad \{2, 4\} \quad \{3, 4\}$
 $\{1, 5\} \quad \{2, 5\} \quad \{3, 5\} \quad \{4, 5\}$

10 subsets

3-element subsets:
 $\{1, 2, 3\}$
 $\{1, 2, 4\} \quad \{1, 3, 4\} \quad \{2, 3, 4\}$
 $\{1, 2, 5\} \quad \{1, 3, 5\} \quad \{1, 4, 5\} \quad \{2, 3, 5\} \quad \{2, 4, 5\} \quad \{3, 4, 5\}$

10 subsets

4-element subsets: $\{1, 2, 3, 4\}, \{1, 2, 3, 5\}, \{1, 2, 4, 5\}, \{1, 3, 4, 5\}, \{2, 3, 4, 5\}$

5 subsets

5-element subsets: $\{1, 2, 3, 4, 5\}$

1 subset

4. a) $\{1\}$ b) $\{1, 4, 5, 6, 7\}$ c) $\{3, 8\}$ d) $\{2, 4, 5, 6, 7, 8, 9\}$ e) $\{2, 4, 5, 6, 7, 8, 9\}$

f) $\{1, 3, 4, 5, 6, 7, 8, 9\}$ g) $\{5, 7, 9\}$ h) 7

5. a) true b) true c) false d) true e) true f) true g) true h) false i) false j) true k) true l) true

6. $\{1, 4\}$

$\{1, 5\} \quad \{4, 5\}$

$\{1, 6\} \quad \{4, 6\} \quad \{5, 6\}$

$\{1, 7\} \quad \{4, 7\} \quad \{5, 7\} \quad \{6, 7\}$

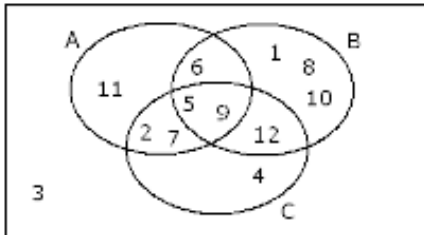
$\{1, 9\} \quad \{4, 9\} \quad \{5, 9\} \quad \{6, 9\} \quad \{7, 9\}$

7. $A \times B = \{(1, p), (2, p), (3, p), (4, p), (1, q), (2, q), (3, q), (4, q)\}$

and

$B \times A = \{(p, 1), (p, 2), (p, 3), (p, 4), (q, 1), (q, 2), (q, 3), (q, 4)\}$

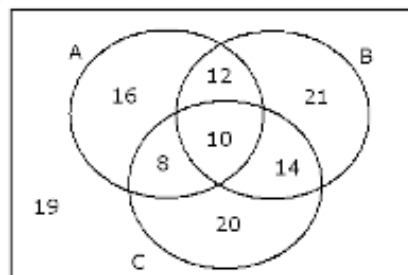
8. a) U



b) $\{1, 4, 8, 10, 12\}$ c) $\{4, 6, 11, 12\}$

d) $\{1, 3, 6, 8, 10, 11, 12\}$ e) true 9. 6 10. 5

11. a)



b) 44 c) 19 d) 85 e) 12 f) 34