Factoring by completing the square is a powerful and elegant factoring technique. We will see later that this is the only method that does not break down once numbers stop being "nice". the square only works if we factor out the leading coeffecient. For now, they will coincide with the greatest common factor.

Example 1. Factor $18x - 3x^2 + 165$ by completing the square.
Step 1. We re-arrange the terms by decreasing order of degree.

$$18x - 3x^2 + 165 = -3x^2 + 18x + 165$$

Step 2. We factor out the greatest common factor.

$$-3x^2 + 18x + 165 = -3\left(x^2 - 6x - 55\right)$$

Step 3. We factor the expression within the parentheses by completing the square.

$-3\left(x^2 - 6x - 55\right) =$ $\qquad (x-3)^2 = x^2 - 6x + 9$

$$\begin{aligned} &= -3\left(\underbrace{x^2 - 6x + 9} - 9 - 55\right) \\ &= -3\left((x-3)^2 - 64\right) \\ &= -3\left((x-3)^2 - 8^2\right) \\ &= -3(x-3+8)(x-3-8) = \boxed{-3(x+5)(x-11)} \end{aligned}$$

Step 4. We check our result by multiplication.

$$-3(x+5)(x-11) = -3\left(x^2 - 11x + 5x - 55\right) = -3\left(x^2 - 6x - 55\right) = -3x^2 + 18x + 165$$

Thus our result, $-3(x+5)(x-11)$ is correct.

Example 2. Factor $267x^2 - 48x^3 + 3x^4$ by completing the square.

$$\begin{aligned} &= 267x^2 - 48x^3 + 3x^4 && \text{rearrange terms} \\ &= 3x^4 - 48x^3 + 267x^2 && \text{factor out } 3x^2 \\ &= 3x^2\left(x^2 - 16x + 89\right) && (x-8)^2 = x^2 - 16x \boxed{+64} \\ &= 3x^2\left(\underbrace{x^2 - 16x + 64} - 64 + 89\right) = && \text{realize comple square, combine like terms} \\ &= 3x^2\left((x-8)^2 + 25\right) \end{aligned}$$

We can not apply the difference of squares theorem, since 25 is added, not subtracted. The sum of squares does not factor.

Thus the expression $\boxed{3x^2\left(x^2 - 16x + 89\right)}$ is completely factored.

Example 3. Factor $5x^2 - 240x + 2160$ by completing the square.

$$5x^2 - 240x + 2160 = \qquad \text{factor out } 5$$

$$\begin{aligned}
&= 5\left(x^2 - 48x + 432\right) && (x-24)^2 = x^2 - 48x \boxed{+576} \\
&= 5\left(\underbrace{x^2 - 48x + 576}{} - 576 + 432\right) \\
&= 5\left((x-24)^2 - 144\right) \\
&= 5\left((x-24)^2 - 12^2\right) \\
&= 5(x - 24 + 12)(x - 24 - 12) \\
&= \boxed{5(x-12)(x-36)}
\end{aligned}$$

We check: $5(x-12)(x-36) = 5\left(x^2 - 12x - 36x + 432\right) = 5\left(x^2 - 48x + 432\right) = 5x^2 - 240x + 2160.$
Thus our result, $5(x-12)(x-36)$ is correct.

Example 4. Factor $9 - y^2 - 8y$ by completing the square.
We first rearrange the terms be degree and then factor out the leading coefficient.

$$9 - y^2 - 8y =$$

$$\begin{aligned}
&= -y^2 - 8y + 9 && (y+4)^2 = y^2 + 8y \boxed{+16} \\
&= -1\left(y^2 + 8y - 9\right) \\
&= -\left(\underbrace{y^2 + 8y + 16}{} - 16 - 9\right) && \text{realize comple square, combine like terms} \\
&= -\left((y+4)^2 - 25\right) && \text{re-write 25 as a square} \\
&= -\left((y+4)^2 - 5^2\right) && \text{factor via the difference of squares theorem} \\
&= -(y + 4 + 5)(y + 4 - 5) && \text{combine like terms, drop extra parentheses} \\
&= \boxed{-(y+9)(y-1)}
\end{aligned}$$

We check: $-(y+9)(y-1) = -\left(y^2 - y + 9y - 9\right) = -\left(y^2 + 8y - 9\right) = -y^2 - 8y + 9$
Thus our result, $-(y+9)(y-1)$ is correct.

Example 5. One number is 12 less than three times the other. Find these numbers if their product is 135.

Solution: Let us denote one number by x. Then the other number is $3x - 12$. The equation will express the product of the numbers.

$$\begin{aligned}
x(3x-12) &= 135 && \text{multiply} \\
3x^2 - 12x &= 135 && \text{subtract } 135 \\
3x^2 - 12x - 135 &= 0
\end{aligned}$$

$$\begin{aligned}
3x^2-12x-135 &= 0 && \text{factor out 3}\\
3\left(x^2-4x-45\right) &= 0 && \text{factor by completing the square}\\
3\left(x^2-4x-45\right) &= 0 && (x-2)^2 = x^2-4x+4\\
3\left(\underbrace{x^2-4x+4}-4-45\right) &= 0 &&\\
3\left((x-2)^2-49\right) &= 0 && 7^2=49\\
3\left((x-2)^2-7^2\right) &= 0 &&\\
3(x-2+7)(x-2-7) &= 0 &&\\
3(x+5)(x-9) &= 0 &&
\end{aligned}$$

$$x_1=-5 \qquad x_2=9$$

We did not get one pair! We obtained two candidates for the number we denoted by x. If the one number is -5, then the other is $3(-5)-12=-27$. If one number is 9, then the other one is $3(9)-12=15$. It is easy to see that both pairs work,

$$\begin{aligned}
-3(-5)-12 &= -27 \quad \text{and} \quad -5(-27)=135\\
3\cdot 9-12 &= 15 \quad \text{and} \quad 9(15)=135
\end{aligned}$$

Thus there are two solutions, $\boxed{-5 \text{ with } -27 \text{ and } 9 \text{ with } 15}$.

Example 6. We throw an object upward from the top of a 1200 ft tall building. The height of the object, (measured in feet) t seconds after we threw it is $h=-16t^2+160t+1200$.

a) Where is the object 3 seconds after we threw it?

b) How long does it take for the object to hit the ground?

Solution: a) We need to compute h when $t=3$. This means that we substitute 3 into t and evaluate the algebraic expression.

$$\begin{aligned}
h &= -16\cdot 3^2+160\cdot 3+1200 = -16\cdot 9+160\cdot 3+1200\\
&= -144+480+1200 = 336+1200 = 1536
\end{aligned}$$

Thus the object is $\boxed{\text{at a height of } 1536\text{ ft}}$ after 3 seconds.

b) To figure out how long it takes for the object to hit the ground, we solve the equation $t=?$ so that $h=0$

$$\begin{aligned}
h &= 0\\
-16t^2+160t+1200 &= 0 \quad \text{factor out } -16\\
-16\left(t^2-10t-75\right) &= 0
\end{aligned}$$

We will factor $t^2-10t+75$ by completing the square.

$$\begin{aligned}
-16\left(t^2-10t-75\right) &= 0 && (t-5)^2=t^2-10t+25 \quad \text{smuggle in 25}\\
-16\left(\underbrace{t^2-10t+25}-25-75\right) &= 0 &&\\
-16\left((t-5)^2-100\right) &= 0 && \text{re-write 100 as } 10^2\\
-16\left((t-5)^2-10^2\right) &= 0 && \text{factor via the difference of squares theorem}\\
-16(t-5+10)(t-5-10) &= 0 && \text{simplify}\\
-16(t+5)(t-15) &= 0 && \text{apply zero property}
\end{aligned}$$

$$
\begin{aligned}
t+5 &= 0 \quad \text{or} \quad t-15=0 \\
t &= -5 \quad \text{or} \quad t=15
\end{aligned}
$$

Since the negative solution, $t=-5$ does not make sense in the context of the problem, it is ruled out. We check. If $t=15$,

$$
\begin{aligned}
h(3) &= -16\cdot 15^2+160\cdot 15+1200 = -16\cdot 225+160\cdot 15+1200 \\
&= -3600+2400+1200 = -1200+1200 = 0 \quad \checkmark
\end{aligned}
$$

Thus the answer is: it will take $\boxed{15 \text{ seconds}}$. for the object to hit the ground.

Practice Problems

Completely factor each of the following by completing the square.

1. $4x+2x^2-30$
2. $70a^2-255a+5a^3$
3. $78b^2-30b^3+3b^4$
4. $32x+2x^2-594$
5. $18c-24c^2+6c^3$
6. $-2d-2d^2-d^3$
7. $432-x^2-6x$
8. $x^2-14x+58$
9. $10abc-600ac+5ab^2c$
10. $70y^3+24y^4+2y^5$
11. $18x^2y^2-216x^2y+3x^2y^3$
12. $1000x-50x^2-5x^3$
13. One number is 20 less than five times another. Find these numbers if their product is 825.
14. We throw an object upward from the top of a 320 ft tall building. The height of the object, (measured in feet) t seconds after we threw it is $h=-16t^2+128t+320$. How long does it take for the object to hit the ground?
15. Four times a number is 21 less than the square of the number. Find this number.

Answers

1. $2(x+5)(x-3)$ 2. $5a(a-3)(a+17)$ 3. $3b^2\left(b^2-10b+26\right)$ 4. $2(x+27)(x-11)$

5. $6c(c-1)(c-3)$ 6. $-d\left(d^2+2d+2\right)$ 7. $-(x+24)(x-18)$ 8. can not be factored

9. $5ac(b+12)(b-10)$ 10. $2y^3(y+7)(y+5)$ 11. $3x^2y(y+12)(y-6)$ 12. $-5x(x+20)(x-10)$

13. -11 with -75 and 15 with 55   14. 10 seconds   15. -3 and 7

