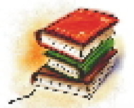
The following problems can all be solved using the factoring techniques we have so far: factoring out the greatest common factor, factoring via the difference of squares theorem, and by completing the square. Often we will need more than one technique in an exercise.



Sample Problems

1. Completely factor each of the following.

   a) $3a^3 - 27ab^2$ c) $2p^4 - 162$ e) $357ab^2 - 30ab^2x - 3ab^2x^2$

   b) $100x - x^2 - 2419$ d) $x^2 - 4x + 7$ f) $20x + 5x^3$

2. Solve each of the following equations. Make sure to check your solution.

   a) $8x^3 = 50x^2$ b) $8p^3 = 50p$ c) $2x^3 = 20x^2 + 1750x$

3. Word Problems

   a) One number is 18 less than the other. Find these numbers if their product is 1600.

   b) One number is 12 less than three times another number. Find these numbers if their product is.135 .

   c) We throw an object upward from the top of a 1200 ft tall building. The height of the object, (measured in feet) t seconds after we threw it is $h(t) = -16t^2 + 160t + 1200$

      i) Where is the object 3 seconds after we threw it?

      ii) How long does it take for the object to hit the ground?

   d) Find all numbers that satisfy the following condition: if we square the number, we get back the same number.

   e) Find all numbers that satisfy the following condition: if we raise the number to the third power, the result is four times the original number.

   f) The area of a rectangle is $1260\ \mathrm{m}^2$. Find the dimensions of the rectangle if we know that one side is $48\ \mathrm{m}$ longer than three times the other side.

Practice Problems

1. Completely factor each of the following.

   a) $a^4 - 16$ d) $36x^2y^3 + 4x^4y^3$ g) $x^2 + 16x + 73 =$

   b) $3p^2 - 12p - 288$ e) $-2x^4 + 162$

   c) $600ab^2 - 6ab^4$ f) $5a^3b^2 - 15ab$

2. Solve each of the following equations. Make sure to check your solutions.

   a) $3x^3 = 75x$ b) $x^3 - 270x = 3x^2$ c) $(x+1)(1-2x) = -3x^2 + 7x + 34$

3. Word Problems.

a) One number is 32 less than the other. Find these numbers if their product is -135.

b) One number is 20 less than twice another number. Find these numbers if their product is 288.

c) We throw an object upward from the top of a $112\,\mathrm{ft}$ tall building. The height of the object, (measured in feet) t seconds after we threw it is

$$h(t) = -16t^2 + 96t + 112$$

i) Where is the object 3 seconds after we threw it?

ii) How long does it take for the object to hit the ground?

d) The area of a rectangle is $1100\,\mathrm{m}^2$. Find the dimensions of the rectangle if we know that one side is $60\,\mathrm{m}$ shorter than five times another side.

Answers

Sample Problems

1. a) $3a(a+3b)(a-3b)$ b) $-(x-41)(x-59)$ c) $2\left(p^2+9\right)(p+3)(p-3)$
   d) cannot be factored over the real numbers e) $-3ab^2(x+17)(x-7)$ f) $5x\left(x^2+4\right)$

2. a) 0 and $\dfrac{25}{4}$ b) $-\dfrac{5}{2}$, 0, and $\dfrac{5}{2}$ c) $35, 0, -25$

3. a) $32, 50$ and $-50, -32$ c) -5 with -27 and 9 by 15 d) i) 1536 ft ii) 15 seconds e) $0, 1$
   f) $0, 2, -2$ g) 14 m by 90 m

Practice Problems

1. a) $(a-2)(a+2)\left(a^2+4\right)$ b) $3(p+8)(p-12)$ c) $-6ab^2(b-10)(b+10)$
   d) $4x^2y^3\left(x^2+9\right)$ e) $-2\left(x^2+9\right)(x+3)(x-3)$ f) $5ab\left(a^2b-3\right)$
   g) cannot be factored over the real numbers

2. a) $-5, 0, 5$ b) $18, 0, -15$ c) $-3, 11$

3. a) -5 with 27 and -27 with 5 b) 16 with 18 and -8 with -36 c) i) 256 ft ii) 7 seconds
   d) $22\,\mathrm{m}$ by $50\,\mathrm{m}$

Sample Problems Solutions

1. Completely factor each of the following.

a) $3a^3 - 27ab^2$

Solution: We start with the greatest common factor (or GCF).

$$\begin{aligned} 3a^3 - 27ab^2 &= && \text{factor out GCF} \\ 3a\left(a^2 - 9b^2\right) &= && \text{re-write } 9b^2 \text{ as } (3b)^2 \\ 3a\left(a^2 - (3b)^2\right) &= && \text{factor via the difference of squares theorem} \\ &= 3a(a+3b)(a-3b) \end{aligned}$$

We check by multiplication:

$$3a(a+3b)(a-3b) = 3a\left(a^2 - 3ab + 3ab - 9b^2\right) = 3a\left(a^2 - 9b^2\right) = 3a^3 - 27ab^2$$

Thus our solution, $3a(a+3b)(a-3b)$ is correct.

b) $100x - x^2 - 2419$

Solution: We rearrange the polynomial by degree of terms. Since it is quadratic with three terms, it may factor by completing the square. Then we need to factor out -1 to work with a leading coefficient 1 within the parentheses.

$$100x - x^2 - 2419 =$$

$$\begin{aligned} &= -x^2 + 100x - 2419 \\ &= -\left(x^2 - 100x + 2419\right) \qquad (x-50)^2 = x^2 - 100x + 2500 \\ &= -\left(\underbrace{x^2 - 100x + 2500} - 2500 + 2419\right) \\ &= -\left((x-50)^2 - 81\right) \\ &= -\left((x-50)^2 - 9^2\right) \\ &= -(x - 50 + 9)(x - 50 - 9) \\ &= -(x-41)(x-59) \end{aligned}$$

We check by multiplication:

$$-(x-41)(x-59) = -\left(x^2 - 41x - 59x + 2419\right) = -\left(x^2 - 100x + 2419\right) = -x^2 + 100x - 2419$$

c) $2p^4 - 162$

Solution: We start with the greatest common factor (or GCF).

$$\begin{aligned}
2p^4 - 162 &= && \text{factor out GCF} \\
2\left(p^4 - 81\right) &= && \text{re-write both quantities as squares} \\
2\left(\left(p^2\right)^2 - 9^2\right) &= && \text{factor via the difference of squares theorem} \\
2\left(p^2 + 9\right)\left(p^2 - 9\right) &= && \text{second factor will factor again} \\
2\left(p^2 + 9\right)\left(p^2 - 3^2\right) &= && \text{factor via the difference of squares theorem} \\
&= 2\left(p^2 + 9\right)(p + 3)(p - 3)
\end{aligned}$$

We check by multiplication:

$$\begin{aligned}
2\left(p^2 + 9\right)\underbrace{(p + 3)(p - 3)}_{\text{FOIL}} &= 2\left(p^2 + 9\right)\left(p^2 - 3p + 3p - 9\right) = 2\underbrace{\left(p^2 + 9\right)\left(p^2 - 9\right)}_{\text{FOIL}} \\
&= 2\left(p^4 - 9p^2 + 9p^2 - 81\right) = 2\left(p^4 - 81\right) = 2p^4 - 162
\end{aligned}$$

Thus our solution, $2\left(p^2 + 9\right)(p + 3)(p - 3)$ is correct.

d) $x^2 - 4x + 7$

Solution: We complete the square.

$$\begin{aligned}
x^2 - 4x + 7 &= && (x - 2)^2 = x^2 - 4x + 4 \\
\underbrace{x^2 - 4x + 4}\, - 4 + 7 &= (x - 2)^2 + 3
\end{aligned}$$

This expression can not be factored because the sum of squares can not be factored over the real numbers.

e) $357ab^2 - 30ab^2x - 3ab^2x^2$

Solution: We start with the GCF (greatest common factor) and rearrange the polynomial by degree of terms.

$$\begin{aligned}
357ab^2 - 30ab^2x - 3ab^2x^2 &= 3ab^2\left(119 - x^2 - 10x\right) \\
&= 3ab^2\left(-x^2 - 10x + 119\right)
\end{aligned}$$

The expression is quadratic with three terms, and so it may factor by completing the square. Then we need to factor out -1 to make the leading coefficient 1.

$$\begin{aligned}
&357ab^2 - 30ab^2x - 3ab^2x^2 = \\
&= 3ab^2\left(119 - x^2 - 10x\right) \\
&= -3ab^2\left(x^2 + 10x - 119\right) && (x + 5)^2 = x^2 + 10x + 25 \\
&= -3ab^2\left(\underbrace{x^2 + 10x + 25}\, - 25 - 119\right) \\
&= -3ab^2\left((x + 5)^2 - 144\right) \\
&= -3ab^2\left((x + 5)^2 - 12^2\right) \\
&= -3ab^2(x + 5 + 12)(x + 5 - 12) \\
&= -3ab^2(x + 17)(x - 7)
\end{aligned}$$

We can check by multiplication.

f) $20x + 5x^3$

Solution: We rearrange the terms by degree first and then factor out the GCF.

$$20x + 5x^3 = 5x^3 + 20x = 5x\left(x^2 + 4\right)$$

Since the sum of squares can not be factored, the final answer is $5x\left(x^2 + 4\right)$. We can easily check the result by mulitplication.

2. Solve each of the following equations. Make sure to check your solution.

a) $8x^3 = 50x^2$

Solution: since this equation is of a higher degree than 1, our only method is to reduce one side to zero, factor, and then apply the zero poduct rule.

$$\begin{aligned} 8x^3 &= 50x^2 && \text{subtract } 50x^2 \\ 8x^3 - 50x^2 &= 0 && \text{the GCF is } 2x^2 \\ 2x^2(4x - 25) &= 0 \end{aligned}$$

We now apply the zero product rule. If this product is zero, then either $2x^2 = 0$ or $4x - 25 = 0$. We solve these equations for x.

$$\begin{aligned} 2x^2 &= 0 && \text{or} && 4x - 25 = 0 \\ 2 \cdot x \cdot x &= 0 && \text{or} && 4x = 25 \\ x &= 0 && \text{or} && x = \frac{25}{4} \end{aligned}$$

We check both solutions. If $x = 0$, then

$$\begin{aligned} \text{LHS} &= 8 \cdot 0^3 = 8 \cdot 0 = 0 \\ \text{RHS} &= 50 \cdot 0^2 = 50 \cdot 0 = 0 \end{aligned}$$

If $x = \frac{25}{4}$, then

$$\begin{aligned} \text{LHS} &= 8\left(\frac{25}{4}\right)^3 = \frac{8}{1} \cdot \frac{15\,625}{64} = \frac{15\,625}{8} \\ \text{RHS} &= 50\left(\frac{25}{4}\right)^2 = \frac{50}{1} \cdot \frac{625}{16} = \frac{15\,625}{8} \end{aligned}$$

Thus both solutions, 0 and $\frac{25}{4}$ are correct.

b) $8p^3 = 50p$

Solution: since this equation is of a higher degree than 1, our only method is to reduce one side to zero, factor, and then apply the zero product rule.

$$\begin{aligned} 8p^3 &= 50p && \text{subtract } 50p \\ 8p^3 - 50p &= 0 && \text{the GCF is } 2p \\ 2p\left(4p^2-25\right) &= 0 \\ 2p\left((2p)^2-5^2\right) &= 0 && \text{factor via difference of squares theorem} \\ 2p(2p+5)(2p-5) &= 0 \end{aligned}$$

We now apply the zero product rule. If this product is zero, then either $2p = 0$ or $2p + 5 = 0$ or $2p - 5 = 0$. We solve these equations for p.

$$\begin{aligned} 2p+5 &= 0 && \text{or} && 2p-5=0 && \text{or} && 2p=0 \\ 2p &= -5 && \text{or} && 2p=5 && \text{or} && p=0 \\ p &= -\frac{5}{2} && \text{or} && p=\frac{5}{2} \end{aligned}$$

We check all three solutions. If $p = -\dfrac{5}{2}$, then

$$\text{LHS} = 8\left(-\frac{5}{2}\right)^3 = \frac{8}{1}\cdot\frac{-125}{8} = -125 \quad \text{and} \quad \text{RHS} = 50\left(-\frac{5}{2}\right) = \frac{50}{1}\cdot\frac{-5}{2} = \frac{-250}{2} = -125$$

If $p = \dfrac{5}{2}$, then

$$\text{LHS} = 8\left(\frac{5}{2}\right)^3 = \frac{8}{1}\cdot\frac{125}{8} = 125 \quad \text{and} \quad \text{RHS} = 50\left(\frac{5}{2}\right) = \frac{50}{1}\cdot\frac{5}{2} = \frac{250}{2} = 125$$

and if $p = 0$, then $\text{LHS} = 8\cdot 0^3 = 8\cdot 0 = 0$ and $\text{RHS} = 50\cdot 0 = 0$

Thus all three solutions, $-\dfrac{5}{2}$, 0, and $\dfrac{5}{2}$ are correct.

c) $2x^3 = 20x^2 + 1750x$

Solution: We reduce one side to zero, then factor, and then apply the zero property.

$$\begin{aligned} 2x^3 &= 20x^2 + 1750x \\ 2x^3 - 20x^2 - 1750x &= 0 && \text{factor out GCF} \\ 2x\left(x^2-10x-875\right) &= 0 && \text{divide both sides by 2} \\ x\left(x^2-10x-875\right) &= 0 \end{aligned}$$

We will factor by completing the square. Half of the linear coefficient is -5, and thus we will work with $(x-5)^2 = x^2 - 10x + 25$. We smuggle in 25.

$$\begin{aligned} x\left(x^2-10x-875\right) &= 0 \\ x\left(\underbrace{x^2-10x+25}-25-875\right) &= 0 \\ x\left((x-5)^2-900\right) &= 0 \end{aligned}$$

$$\begin{aligned} x\left((x-5)^2-30^2\right) &= 0 \\ x(x-5+30)(x-5-30) &= 0 \\ x(x+25)(x-35) &= 0 \\ x = 0 \text{ or } -25, &\text{ or } 35 \end{aligned}$$

We check (even if it hurts a little...). If $x=0$, then

$$\text{LHS}=2(0)^3=0 \text{ and } \text{RHS}=20(0)^2+1750(0)=0$$

If $x=-25$, then

$$\begin{aligned}\text{LHS} &= 2(-25)^3=2(-15\,625)=-31\,250\\ \text{RHS} &= 20(-25)^2+1750(-25)=20(625)+1750(-25)=12\,500-43\,750=-31\,250\end{aligned}$$

And if $x=35$, then

$$\begin{aligned}\text{LHS} &= 2(35)^3=2(42\,875)=85\,750\\ \text{RHS} &= 20(35)^2+1750(35)=20(1225)+1750(35)=24\,500+61\,250=85\,750\end{aligned}$$

3. Word Problems

a) One number is 18 less than the other. Find these numbers if their product is 1600.

Solution: Let us denote the smaller number by x. Then the larger number is $x+8$. The equation will express the product of the numbers.

$$\begin{aligned}x(x+18) &= 1600 && \text{multiply}\\ x^2+18x &= 1600 && \text{subtract 1600}\\ x^2+18x-1600 &= 0 && \text{factor by completing the square}\\ x^2+18x-1600 &= 0 && (x+9)^2=x^2+18x+81\\ \underbrace{x^2+18x+81}-81-1600 &= 0 &&\\ (x+9)^2-1681 &= 0 && \sqrt{1681}=41\\ (x+9)^2-41^2 &= 0 &&\\ (x+9+41)(x+9-41) &= 0 &&\\ (x+50)(x-32) &= 0 &&\end{aligned}$$

$$x_1=-50 \qquad x_2=32$$

We did not get one pair! We obtained two candidates for the smaller number in two pairs of numbers. If the smaller number is -50, then the larger one is $-50+18=-32$. If the smaller number is 32, then the larger one is $32+18=50$. It is easy to see that both pairs work:

$$\begin{aligned}-32-(-50) &= 18 \quad\text{and}\quad -32(-50)=1600\\ 50-32 &= 18 \quad\text{and}\quad 32(50)=1600\end{aligned}$$

Thus there are two solutions, -50 with -32 and 32 with 50.

b) One number is 12 less than three times the other. Find these numbers if their product is 135.

Solution: Let us denote one number by x. Then the other number is $3x-12$. The equation will express the product of the numbers.

$$\begin{aligned}
x(3x-12) &= 135 && \text{multiply} \\
3x^2-12x &= 135 && \text{subtract } 135 \\
3x^2-12x-135 &= 0 && \text{factor out } 3 \\
3\left(x^2-4x-45\right) &= 0 && \text{factor by completing the square} \\
3\left(x^2-4x-45\right) &= 0 && \quad (x-2)^2 = x^2-4x+4 \\
3\left(\underbrace{x^2-4x+4}-4-45\right) &= 0 && \\
3\left((x-2)^2-49\right) &= 0 && \qquad 7^2=49 \\
3\left((x-2)^2-7^2\right) &= 0 && \\
3(x-2+7)(x-2-7) &= 0 && \\
3(x+5)(x-9) &= 0 &&
\end{aligned}$$

$$x_1 = -5 \qquad x_2 = 9$$

We did not get one pair! We obtained two candidates for the number we denoted by x. If the one number is -5, then the other is $3(-5)-12=-27$. If one number is 9, then the other one is $3(9)-12=15$. It is easy to see that both pairs work,

$$\begin{aligned}
-3(-5)-12 &= -27 \quad \text{and} \quad -5(-27)=135 \\
3\cdot 9-12 &= 15 \quad \text{and} \quad 9(15)=135
\end{aligned}$$

Thus there are two solutions, -5 with -27 and 9 with 15.

c) We throw an object upward from the top of a 1200 ft tall building. The height of the object, (measured in feet) t seconds after we threw it is

$$h(t) = -16t^2+160t+1200$$

i) Where is the object 3 seconds after we threw it?

Solution: We need to compute $h(3)$. This means that we substitute 3 into t and evaluate the algebraic expression.

$$\begin{aligned}
h(3) &= -16\cdot 3^2+160\cdot 3+1200 = -16\cdot 9+160\cdot 3+1200 \\
&= -144+480+1200 = 336+1200 = 1536
\end{aligned}$$

Thus the object is at a height of 1536 ft after 3 seconds.

ii) How long does it take for the object to hit the ground?

Solution: we need to solve the equation $t = ?$ so that $h(t) = 0$

$$\begin{aligned} h(t) &= 0 \\ -16t^2 + 160t + 1200 &= 0 \quad \text{factor out } -16 \\ -16\left(t^2 - 10t - 75\right) &= 0 \end{aligned}$$

We will factor $t^2 - 10t + 75$ by completing the square.

$$\begin{aligned} -16\left(t^2 - 10t - 75\right) &= 0 && (t-5)^2 = t^2 - 10t + 25 \quad \text{smuggle in } 25 \\ -16\left(\underbrace{t^2 - 10t + 25} - 25 - 75\right) &= 0 \\ -16\left((t-5)^2 - 100\right) &= 0 && \text{re-write } 100 \text{ as } 10^2 \\ -16\left((t-5)^2 - 10^2\right) &= 0 && \text{factor via the difference of squares theorem} \\ -16(t-5+10)(t-5-10) &= 0 && \text{simplify} \\ -16(t+5)(t-15) &= 0 && \text{apply zero property} \end{aligned}$$

$$\begin{aligned} t+5 &= 0 \quad \text{or} \quad t - 15 = 0 \\ t &= -5 \quad \text{or} \quad t = 15 \end{aligned}$$

Since the negative solution, $t = -5$ does not make sense in the context of the problem, it is ruled out. We check $t = 15$:

$$\begin{aligned} h(3) &= -16 \cdot 15^2 + 160 \cdot 15 + 1200 \\ &= -16 \cdot 225 + 160 \cdot 15 + 1200 \\ &= -3600 + 2400 + 1200 \\ &= -1200 + 1200 = 0 \end{aligned}$$

Thus the answer is: 15 seconds.

c) Find all numbers that satisfy the following condition: if we square the number, we get back the same number.

Solution: Let us denote the number by x. The equation is

$$\begin{aligned} x^2 &= x && \text{reduce one side to zero} \\ x^2 - x &= 0 && \text{factor} \\ x(x-1) &= 0 && \text{apply the zero property} \end{aligned}$$

$$\begin{aligned} x &= 0 \quad \text{or} \quad x - 1 = 0 \\ x &= 0 \quad \text{or} \quad x = 1 \end{aligned}$$

Thus there are two numbers, 0 and 1, satisfying the property. We check: $0^2 = 0$ and $1^2 = 1$. Thus our answer is: 0 and 1.

d) Find all numbers that satisfy the following condition: if we raise the number to the third power, the result is four times the original number.

Solution: Let us denote the number by x. The equation is

$$\begin{aligned} x^3 &= 4x && \text{reduce one side to zero} \\ x^3 - 4x &= 0 && \text{factor out the GCF} \\ x\left(x^2-4\right) &= 0 && \text{factor via the difference of squares theorem} \\ x(x+2)(x-2) &= 0 && \text{apply the zero property} \end{aligned}$$

$$\begin{aligned} x &= 0 \quad \text{or} \quad x+2=0 \quad \text{or} \quad x-2=0 \\ x &= 0 \quad \text{or} \quad x=-2 \quad \text{or} \quad x=2 \end{aligned}$$

Thus there are three numbers, 0, 2 and -2, satisfying the property. We check: $0^3 = 4 \cdot 0,\ 2^3 = 4 \cdot 2,$ and $-2^3 = 4(-2)$. Thus our answer is: $0,\ 2,$ and -2.

e) The area of a rectangle is $1260\ \text{m}^2$. Find the dimensions of the rectangle if we know that one side is $48\ \text{m}$ longer than three times the other side.

Solution: Let us denote the shorter side by x. Then the longer side is $3x+48$. We obtain the equation for the area (multiply the two sides):

$$\underbrace{x}_{\text{shorter side}}\underbrace{(3x+48)}_{\text{longer side}} = 1260$$

Since this equation is quadratic, we will reduce one side to zero, and factor the other side to solve the equation.

$$\begin{aligned} x(3x+48) &= 1260 && \text{distribute} \\ 3x^2+48x &= 1260 && \text{subtract 1260} \\ 3x^2+48x-1260 &= 0 && \text{factor out the GCF, 3} \\ 3\left(x^2+16x-420\right) &= 0 && \text{divide by 3} \\ x^2+16x-420 &= 0 && \text{factor} \end{aligned}$$

We will factor by completing the square.

$$\begin{aligned} x^2+16x-420 &= 0 \qquad\qquad (x+8)^2 = x^2+16x+64 \\ \underbrace{x^2+16x+64}-64-420 &= 0 \\ (x+8)^2-484 &= 0 \\ (x+8)^2-22^2 &= 0 \\ (x+8+22)(x+8-22) &= 0 \\ (x+30)(x-14) &= 0 \\ x_1 &= -30 \quad \text{and} \quad x_2 = 14 \end{aligned}$$

Since distances can not be negative, $x=-30$ is ruled out. If $x = 14\ \text{m}$, then the other side is $3(14\,\text{m}) + 48\,\text{m} = 90\,\text{m}$. We check: $90\,\text{m} = 3(14\,\text{m}) + 48\,\text{m}$ and $14\,\text{m}\,(90\,\text{m}) = 1260\,\text{m}^2$. Thus the rectangle's dimensions are indeed $14\,\text{m}$ by $90\,\text{m}$.

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