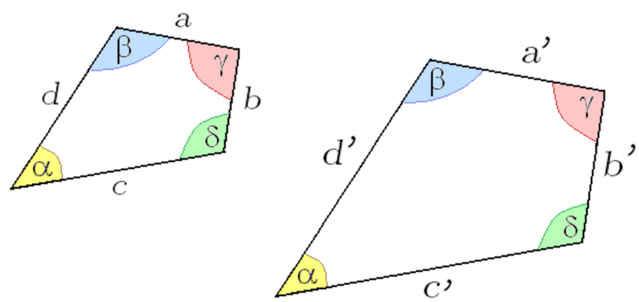
Definition: Two objects are **similar** to each other if they are either identical (also called congruent) or one is an enlargement of the other.

Similar objects always have the following properties: Their corresponding angles are the same, and their corresponding sides are proportional. The two quadrilaterals shown are similar. When labeling the sides, we often suggest which side corresponds to which side. The side corresponding to a is often denoted by a'.

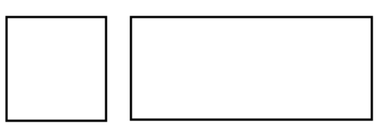


The corresponding sides are proportional. This can be approached in two different ways as follows.

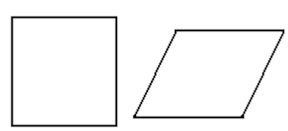
1. We can express how one quadrilateral was enlarged to obtain the other. In this case, every side was doubled. We can express this magnifying factor as the same for all pairs of corresponding sides between the two quadrilaterals.
$$\mu = \frac{a'}{a} = \frac{b'}{b} = \frac{c'}{c} = \frac{d'}{d}$$
2. We can also express that from one quadrilateral to the other, the ratio between corresponding sides was preserved.
$$\frac{a}{b} = \frac{a'}{b'} \text{ and } \frac{a}{c} = \frac{a'}{c'} \text{ and } \frac{b}{c} = \frac{b'}{c'}$$

The two approaches express the same thing. Consider for example $\dfrac{a'}{a} = \dfrac{b'}{b}$. If we clear the denominators by multiplying both sides by ab, then we get $a'b = ab'$. If we start with $\dfrac{a}{b} = \dfrac{a'}{b'}$ and clear the denominators, we also get $ab' = a'b$. Consequently, we are free to express similarity in either way. We often select the form that will lead to easier computation. Most of the time all quantities involved are all positive. This means that we can take the reciprocal of both sides of an equation expressing these ratios without the need to worry about division by zero.

In case of quadrilaterals, they are similar to each other if their sides are proportional **and** have the same angles. Proportional sides alone or equal angles alone can not guarantee similarity.



The square and the rectangle clearly have the same angles and yet the two are not similar. Any enlargement of a square must be a square.



The picture shows a square and a parallelogram with all four of its sides of the same length. Such a quadrilateral is called a *rhombus*. The two are also not similar although they have the same side lengths.

In case of quadrilaterals, to establish similarity, we must prove that the angles are the same and that sides are proportional. Triangles are somehow simpler than quadrilaterals. If we know that two triangles have the same angles, that alone is enough for the two to be similar. If they are similar, then corresponding sides are proportional, therefore we can solve for sides using ratios. This is by far the most frequest use of similarity. We chase down angles to prove similarity so that we can solve for sides.

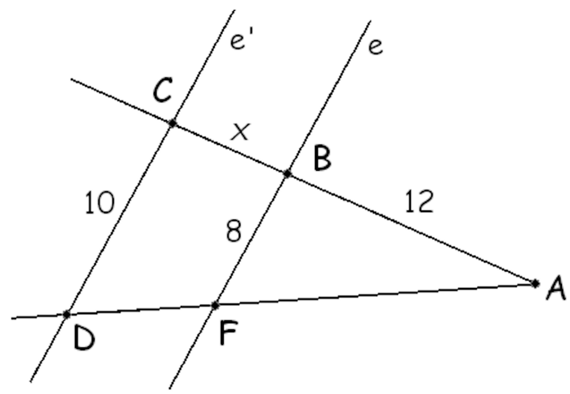
If two triangles have proportional sides, then that alone implies similarity. So now we can conclude that the angles are the same. There are three cases of similarity.

Theorem: Two triangles are similar to each other if

1. they have the same angles, or
2. if all three pairs of sides are proportional, or
3. if two pairs of sides are proportional and the angle enclosed by them are equal.

Most often we establish similarity using angles, and then solve for missing sides using proportionality.

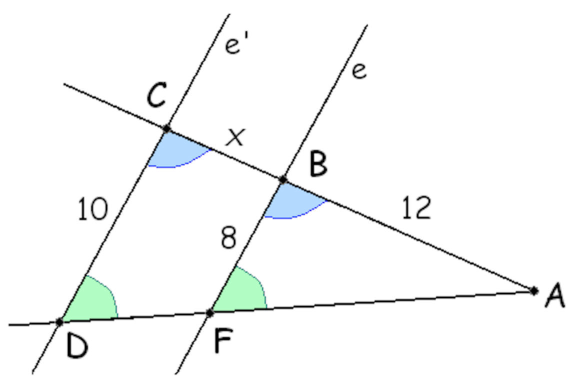
Example 1. Find the value of x based on the figure, given that lines e and e' are parallel.



Solution: First we will use angles to establish that triangles ACD and ABF are similar. Then we use proportonality of sides to find the value of x. Angles ABF and ACD, (marked orange) have equal measures because lines e and e' are parallel. Similarly, angles AFB and ADC, (marked green) have equal measures because lines e and e' are parallel.

The third angle, at point A is shared by the two triangles. (Notice however, that we have similarity with only two pairs of equal angles.) So the two triangles are similar.

Because they are similar, corresponding sides of the two triangles are proportional. We can use this fact to find the value of x. Notice that x, or line segment BC is not a side of any triangles.



It is recommended that we keep track of sides by the angles opposite them. Our eyes seem to work much better with angles than with sides. So we will write the ratio

$$\frac{\text{side opposite the green angle}}{\text{side opposite the angle at point } A}$$

in both triangles. By similarity, this ratio must remain the same.

In triangle ACD, the ratio $\dfrac{\text{side opposite the green angle}}{\text{side opposite the angle at point } A} = \dfrac{x+12}{10}$.

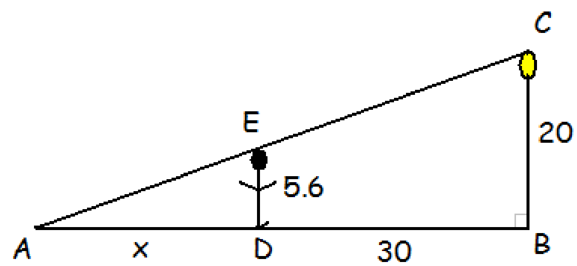
The same ratio in triangle ABF is $\dfrac{12}{8} = \dfrac{3}{2}$. Therefore, we have a linear equation for x.

$$\begin{aligned}
\frac{x+12}{10} &= \frac{3}{2} && \text{multiply by 10}\\
x+12 &= 15 && \text{subtract 12}\\
x &= 3
\end{aligned}$$

Now that we know the value of x, we see how the proportionality works: 15 to 12 is the same as 10 to 8. Therefore, our solution, $\boxed{x=3}$ is correct.

Example 2. A 5.6ft tall person is standing 30ft away from a street light that is 20ft tall. How long is her shadow?

Solution: The shadow is created as light is absorbed by the person, so it can not hit the ground behind her. After we draw a picture, we notice that triangles ABC and ADE are similar. They share an angle at point A, and they both also have a right angle. Because the triangles are similar, corresponding



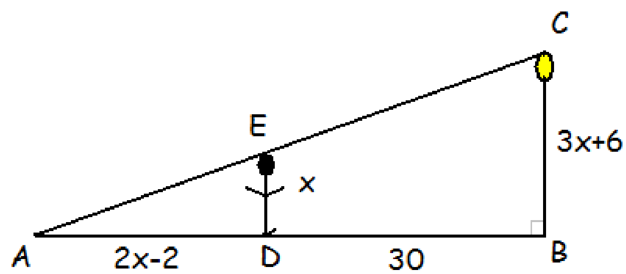
sides are proportional. We can write the ratio $\dfrac{\text{horizontal side}}{\text{vertical side}}$ for both triangles. We need to be careful with the horizontal side of the larger triangle: it is not 30 but rather $x+30$.

$$\begin{aligned}
\frac{x}{5.6} &= \frac{x+30}{20} && \text{mulitply by } 20 \cdot 5.6\\
20x &= 5.6\,(x+30) && \text{multiply by 10 (to get rid of the decimal)}\\
200x &= 56\,(x+30) && \text{distribute 56}\\
200x &= 56x+1680 && \text{subtract } 56x\\
144x &= 1680 && \text{divide by 144}\\
x &= \frac{1680}{144} = \frac{35}{3} = 11\frac{2}{3}
\end{aligned}$$

Thus her shadow is $\boxed{11\frac{2}{3}\text{ft}}$ long.

Example 3. A person is standing 30 feet away from a street light that is six feet taller than three times the height of the person. The length of his shadow is two feet less than twice his height. How tall is he?

Solution: The shadow is created as light is absorbed by the person, so it can not hit the ground behind her. After we draw a picture, we notice that triangles ABC and ADE are similar. They share an angle at point A, and they both also have a right angle.



If we label the person's height by x, then the height of the light is $3x+6$ and the length of his shadow is $2x-2$. Now we can use proportionality sides. We can write the ratio $\dfrac{\text{vertical side}}{\text{horizontal side}}$ for both triangles.

$$\frac{x}{2x-2} = \frac{3x+6}{30+(2x-2)} \quad \text{we solve for } x$$

$$\begin{aligned}
\frac{x}{2x-2} &= \frac{3x+6}{2x+28} \\
\frac{x}{2x-2} &= \frac{3x+6}{2x+28} && \text{multiply by } (2x-2)(2x+28) \\
x(2x+28) &= (2x-2)(3x+6) && \text{expand products} \\
2x^2+28x &= 6x^2+12x-6x-12 && \text{combine like terms} \\
2x^2+28x &= 6x^2+6x-12 && \text{subtract } 2x^2\text{, subtract } 28x \\
0 &= 4x^2-22x-12 && \text{divide by 2} \\
2x^2-11x-6 &= 0
\end{aligned}$$

Since this is a quadratic equation, we need to factor and apply the zero product rule.Numerous factoring techniques can be used.

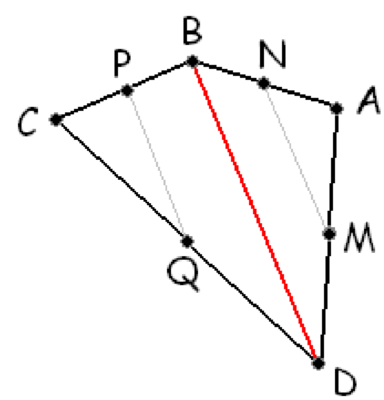
$$(2x+1)(x-6) = 0 \quad \Longrightarrow \quad x_1 = -\frac{1}{2} \text{ and } x_2 = 6$$

Since distances are never negative, $-\dfrac{1}{2}$ is easily ruled out. Thus he is $\boxed{6\text{ft}}$ tall.

The following is an example of how similarity is applied in geometry. Also, it is an example for the third case of similarity: two sides proportional and the angle between them identical.

Example 4. Prove that the midpoints of any convex quadrilateral form a parallelogram.

Solution: Le t $ABCD$ be any convex quadrilateral with midpoints M, N, P, and Q. Notice that AC and BD are the diagonals of quadrilateral $ABCD$.



We first claim that triangles MAN and DAB are similar to each other. Because M and N are midpoints, side AB is twice side AN. Similarly, side AD is twice side AM. The angle between these sides is literally shared. Therefore, triangles MAN and DAB are similar, and all sides of triangle ABC are twice as long as the corresponding sides in triangle ANM. Because of similarity, these triangles have the same angles.
Also, we can see that the corresponding sides are parallel. Therefore, we can state that line sgement BD is parallel to line segment NM, and is twice as long. We have the same situation on the other side, at point C.
Because of P and Q are midpoints, sides PC and CQ are half as long as sides BC and CD. The angle at point C is literally shared,. Therefore, triangles BDC and PCQ are similar. In addition, corresponding sides are parallel. Therefore, line segment BD is parallel to line segment PQ and is twice as long.

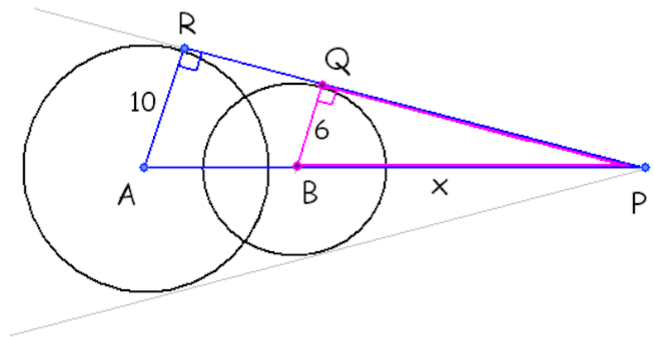
We have therefore that both PQ and NM are line segments parallel to BD, therefore they are parallel to each other. (We also have that PQ and NM are equally long, but we don't need it).

Using an identical argument, we can see that both PN and QM are parallel to line segment AC. therefore, they are parallel to each other. We have that PQ is parallel to NM and PN top QM, therefore $PQMN$ is a parallelogram.

Recall that in circles, the radius drawn to the point of tangency is perpendicular to the tangent line.

Example 5. Suppose that A and B are the centers of two circles with radii 10 units and 6 units long. The distance between A and B is 12 units. The circles have two common tangent lines that intersect each other in point P. Find the distance between the points B and P.

Solution: Let Q and R be the points of tangency as shown on the picture. Triangles APR and BPQ are similar because they share an angle at point P and both have a right angle, at R and Q, correspondingly. If we denote distance PB by x, then



$$\frac{\text{side opposite the right angle}}{\text{side opposite } P} = \frac{x}{6} = \frac{x+12}{10}$$

We solve this equation for x.

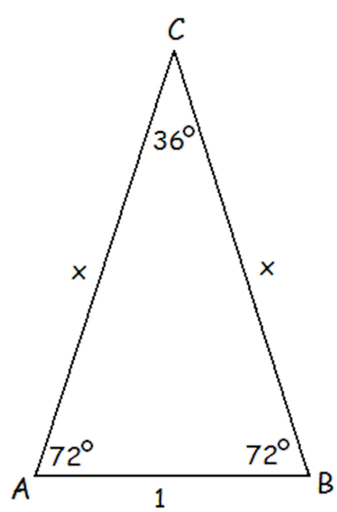
$$\begin{aligned} \frac{x}{6} &= \frac{x+12}{10} \\ 10x &= 6(x+12) \end{aligned} \qquad \begin{aligned} 10x &= 6x+72 \\ 4x &= 72 \\ x &= 18 \end{aligned}$$

So $\overline{PB} = 18$ units.

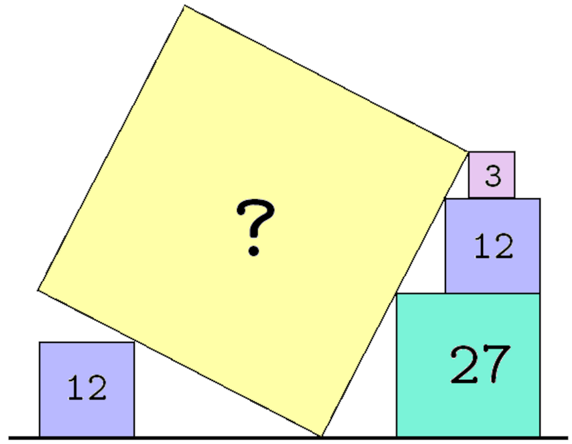
Enrichment

Consider triangle ABC shown on the picture.

1. Bisect the angle at point A. (To bisect an angle means to split it into two equal angles. Such a line is called the angle bisector) This angle bisector intersects side BC in point D. Prove that triangles ABC and ABD are similar.
2. Use the similarity established in the previous problem to find the exact value of x.

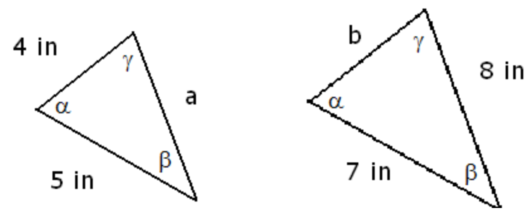


3. This problem is from the great Catriona Shearer. All rectangles on the picture are squares. The numbers show the area of each square. Find the exact value for the area of the yellow square.

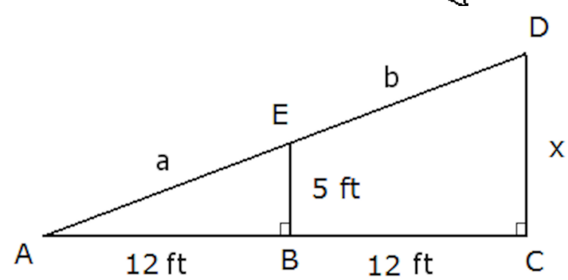


Sample Problems

1. The triangles shown are similar. Find the exact values of a and b shown on the picture below.



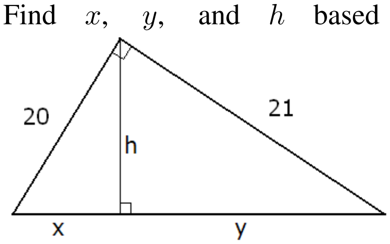
2. Consider the picture shown.
   a) Use the Pythagorean Theorem to find the value of a.
   b) Prove that the triangles ABE and ACD are similar.
   c) Use similar triangles to find the value of x.
   d) Find the value of b.



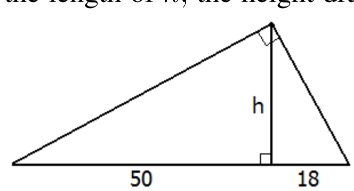
3. a) A person is standing 40ft away from a street light that is 30ft tall. How tall is he if his shadow is 10ft long?
   b) A 6ft tall person is standing 24ft away from a street light that is 15ft tall. How long is her shadow?

4. Prove the following statement. Let ABC be any right triangle, the right angle at point C. The altitude drawn from C to the hypotenuse splits the triangle into two right triangles that are similar to each other and to the original triangle.

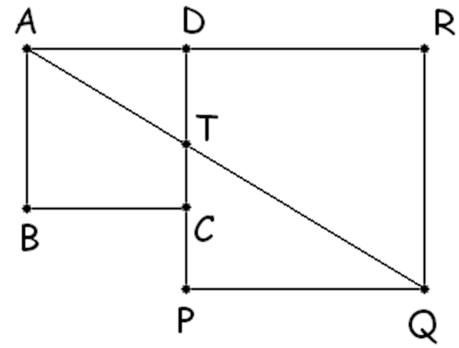
5. Find x, y, and h based on the picture.



6. The picture below shows a right triangle. Find the length of h, the height drawn to the hypotenuse.

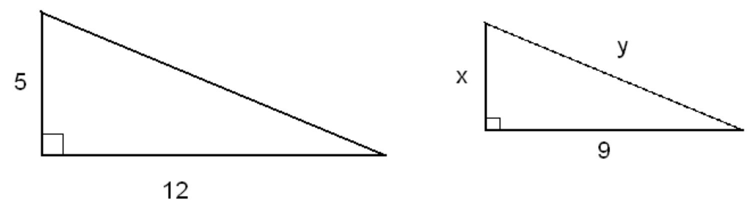


7. Quadrilaterals $ABCD$ and $PQRD$ are both squares. If $AB = 2$ and $PQ = 3$, then find the exact value of the length of TC.

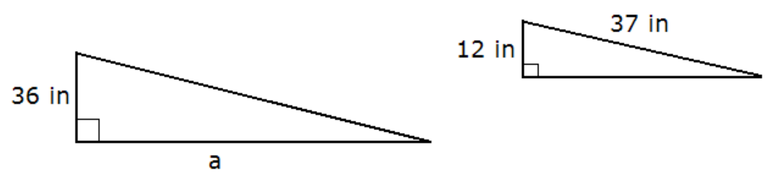


Practice Problems

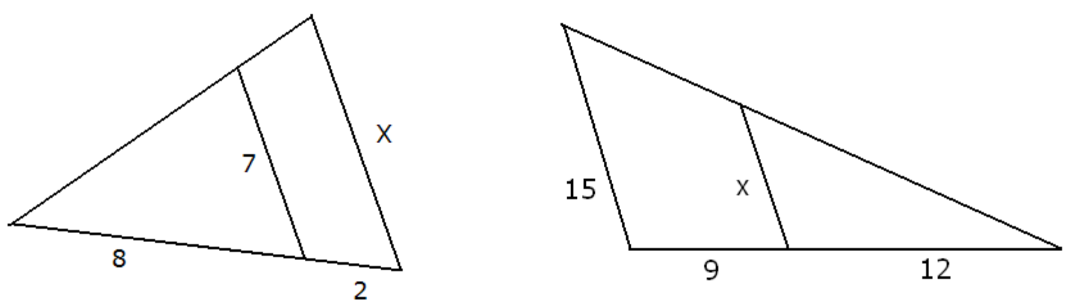
1. The picture shows two similar right triangles. Find the exact values of x and y.



2. The picture shows two similar right triangles. Find the exact value of a.



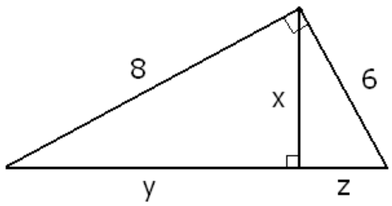
3. Find the value of x based on the figures.



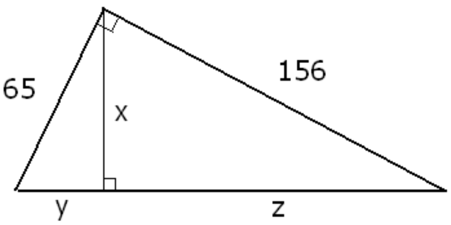
(a) (b)

4. a) A person is standing 24ft away from a street light that is 25ft tall. How tall is he if his shadow is 6ft long?
   b) A 5.2ft tall person is standing 20ft away from a street light that is 15.6ft tall. How long is her shadow?

5. Find the exact value of x, y, and z, based on the figures shown.

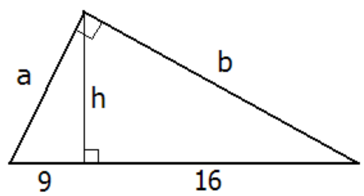


(a)

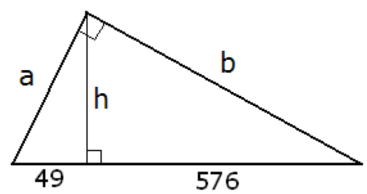


(b)

6. Find the exact value of a, b, and h, based on the picture shown.

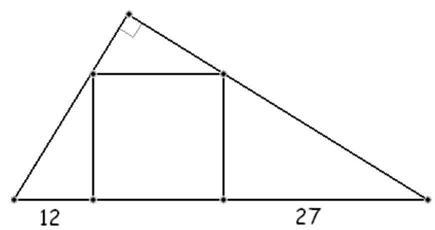


(a)

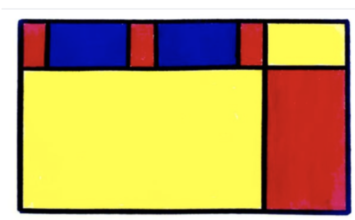


(b)

7. A square is written into a right triangle as shown on the picture. Find the exact value of the length of the side of the square.



8. Another puzzle from Catriona Shearer. All the yellow rectangles are similar to all blue and all red rectangles. What fraction of the design is red?



Answers

Sample Problems

1. $a = \dfrac{40}{7}\text{in},\ \ b = \dfrac{28}{5}\text{in}$ 2. a) 13ft b) see solutions c) 10ft d) 13ft 3. a) 6ft b) 16ft

4. see solutions 5. $h = \dfrac{420}{29},\ \ x = \dfrac{400}{29},\ \ y = \dfrac{441}{29}$ 6. 30 units 7. 1.2 units

Practice Problems

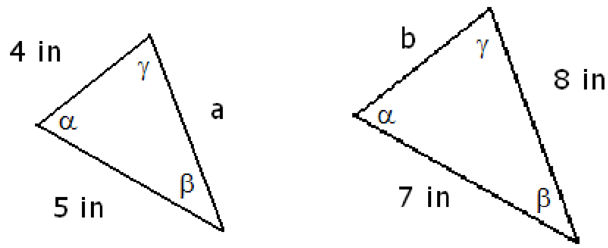
1. $x = \dfrac{15}{4} = 3.75,\ \ y = \dfrac{39}{4} = 9.75$ 2. 105in 3. a) $\dfrac{35}{4} = 8.75$ b) $\dfrac{60}{7}$ 4. a) 5ft b) 10ft

5. a) $x = \dfrac{24}{5} = 4.8,\ \ y = \dfrac{32}{5} = 6.4,\ \ z = \dfrac{18}{5} = 3.6$ b) $x = 60,\ y = 25,\ z = 144$

6. a) $h = 12,\ \ a = 15,\ b = 20$ b) $h = 168,\ a = 175,\ \ b = 600$ 7. 18 units 8. $\dfrac{1}{4}$

Sample Problems Solutions

1. The triangles shown are similar. Find the exact values of a and b shown on the picture.



Solution: In similar triangles, the ratios of corresponding sides are preserved. To find a, we write the ratio $\dfrac{\text{side opposite angle } \alpha}{\text{side opposite angle } \gamma}$ for both triangles.

$$\frac{\text{side opposite angle } \alpha}{\text{side opposite angle } \gamma} = \frac{a}{5} = \frac{8}{7}$$

We now solve the equation for a: $\quad \dfrac{a}{5} = \dfrac{8}{7}$ multiply both sides by 35

$$7a = 40 \qquad \text{divide by 7}$$

$$a = \frac{40}{7}$$

Similarly, we can find b by writing the ratio $\dfrac{\text{side opposite angle } \beta}{\text{side opposite angle } \gamma}$ for both triangles.

$$\frac{\text{side opposite angle } \beta}{\text{side opposite angle } \gamma} = \frac{4}{5} = \frac{b}{7}$$

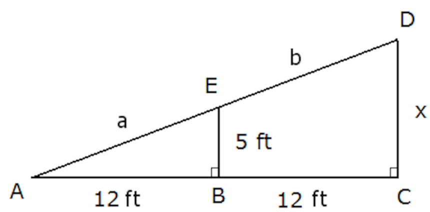
We now solve the equation for b. $\quad \dfrac{4}{5} = \dfrac{b}{7}$ multiply both sides by 35

$$28 = 5b \qquad \text{divide by 5}$$

$$b = \frac{28}{5}$$

Thus $\boxed{a = \frac{40}{7}\text{in and } b = \frac{28}{5}\text{in}}$.

2. Consider the picture shown.



a) Use the Pythagorean Theorem to find the value of a.

Solution: The shorter sides are 5ft and 12ft long. The hypotenuse is a. We state the Pythagorean Theorem for this triangle and solve the equation for a.

$$\begin{aligned} 5^2 + 12^2 &= a^2 \\ 25 + 144 &= a^2 \\ 169 &= a^2 \\ \pm 13 &= a \end{aligned}$$

Since distances can never be negative, $a = -13$ is ruled out. Thus $\boxed{a = 13\text{ft}}$.

b) Prove that the triangles ABE and ACD are similar.

Solution: First, angles ABE and ACD are both right angles. Second, the two triangles literally share angle EAB (or angle DAC). Finally, if two triangles agree in the measure of two of their angles, the third angles must be equal since in every triangle, the three angles add up to $180°$. The two triangles are similar because they have identical angles.

c) Use similar triangles to find the value of x.

Solution: The triangles $\triangle ABE$ and $\triangle ACD$ are similar. To find x, we write the ratio $\dfrac{\text{side opposite point } A}{\text{horizontal side}}$ for both triangles.

$$\frac{\text{side opposite point } A}{\text{horizontal side}} = \frac{5}{12} = \frac{x}{24}$$

and solve the equation for x.

$$\begin{aligned} \frac{5}{12} &= \frac{x}{24} && \text{multiply both sides by 24} \\ 10 &= x \end{aligned}$$

Thus x is $\boxed{10\text{ft}}$. Indeed, once we established that the triangles are similar, and noticed that the horizontal side was doubled from 12ft to 24ft, we could easily predict this answer.

d) Find the value of b.

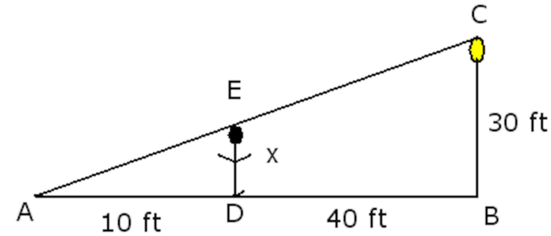
Solution: We can either use similar triangles or the Pythagorean Theorem to find the side AD. Either way, we easily get that 26ft. However, the length of side AD is not b, but $a + b$. From part a), we know that $a = 13\text{ft}$.

$$\begin{aligned} 13 + b &= 26 \\ b &= 13 \end{aligned}$$

Thus $\boxed{b = 13\text{ft}}$.

3. a) A person is standing 40ft away from a street light that is 30ft tall. How tall is he if his shadow is 10ft long?

Solution: After we draw a picture, we see that this problem is very similar to the previous one. Triangles $\triangle ADE$ and $\triangle ABC$ are similar.



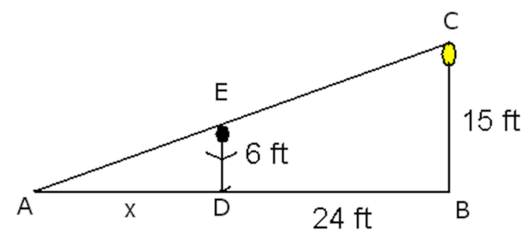
We use the ratio $\dfrac{DE}{AD} = \dfrac{BC}{AB}$ and solve for x.

$$\begin{aligned} \frac{x}{10} &= \frac{30}{50} && \text{multiply both sides by } 50 \\ 5x &= 30 && \text{divide by } 5 \\ x &= 6 \end{aligned}$$

Thus the person is $\boxed{6\text{ft}}$ tall. Notice that the number 40 did not occur in the equation. It is a common error to use 40 instead of 50.

b) A 6ft tall person is standing 24ft away from a street light that is 15ft tall. How long is her shadow?

Solution: After we draw a picture, write an equation expressing that triangles ADE and ABC are similar.



We can use the same ratio as before, $\dfrac{DE}{AD} = \dfrac{BC}{AB}$ and solve for x.

$$\begin{aligned} \frac{6}{x} &= \frac{15}{x+24} && \text{multiply both sides by } x(x+24) \\ 6(x+24) &= 15x && \text{distribute} \\ 6x+144 &= 15x && \text{subtract } 6x \\ 144 &= 9x && \text{divide by } 9 \\ 16 &= x \end{aligned}$$

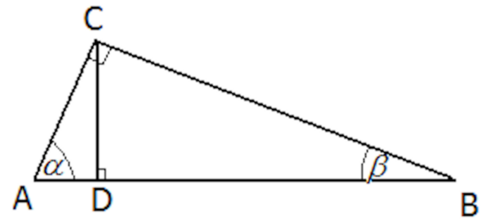
Thus her shadow is $\boxed{16\text{ft}}$ long.

Note: If the first step, multiplying by $x(x+24)$ (same as cross-multiplying) is confusing, here is the break-down:

$$\begin{aligned} \frac{6}{x} &= \frac{15}{x+24} && \text{multiply by } x(x+24) \\ x(x+24)\frac{6}{x} &= \frac{15}{x+24}\cdot x(x+24) && \text{expressing everything as a fraction} \\ \frac{x(x+24)}{1}\cdot\frac{6}{x} &= \frac{15}{x+24}\cdot\frac{x(x+24)}{1} \\ \frac{x(x+24)6}{x} &= \frac{15x(x+24)}{x+24} && \text{cancel} \\ \frac{(x+24)6}{1} &= \frac{15x}{1} && \text{simplify} \\ (x+24)6 &= 15x \end{aligned}$$

4. Prove the following statement. Let ABC be any right triangle, the right angle at point C. The altitude drawn from C to the hypotenuse splits the triangle into two right triangles that are similar to each other and to the original triangle.

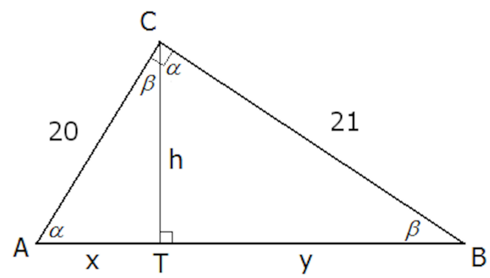
   Solution: Let us draw a picture and use standard labeling of points.



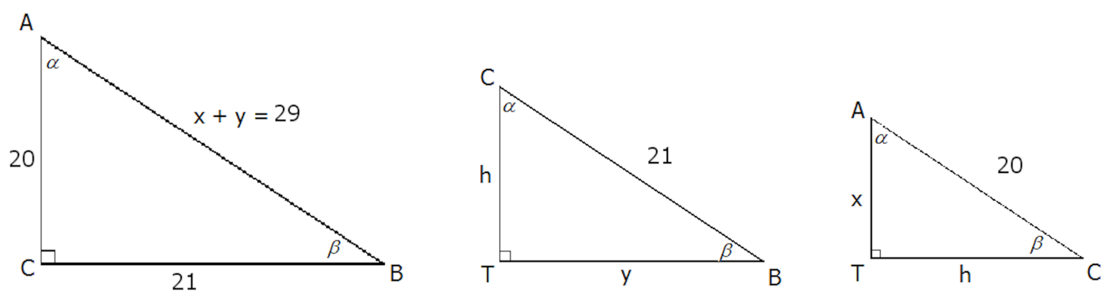
   The two triangles created, $\triangle\ ADC$ and $\triangle\ DBC$ are both right triangles. $\triangle ADC$ is similar to the original triangle, because they agree in two angles: the right angle and α. $\triangle DBC$ is similar to the original triangle, because they agree in two angles: the right angle and β. Thus all three triangles are similar. Also, this will be very useful later: $\angle ACD = \beta$ and $\angle BCD = \alpha$.

5. Find x, y, and h based on the picture.

   Solution: We can easily find the hypotenuse of this triangle via the Pythagorean Theorem. The hypotenuse, $x + y$ is $\sqrt{20^2 + 21^2} = 29$ units long. Next, let us first label the points, angles and sides in the triangle.



   We now re-draw the three similar triangles in a separate figure, all three of them rotated and reflected into the same direction. This way, it is easy to realize what sides correspond to each other. (Hint: start with the anges, they are in the same location. Then identify the points, and finally the sides.)



   We can find y using the following ratio in the first two triangles: $\dfrac{\text{side opposite } \alpha}{\text{hypotenuse}} = \dfrac{21}{29} = \dfrac{y}{21}$

$$\begin{aligned} \frac{21}{29} &= \frac{y}{21} && \text{multiply both sides by } 21 \cdot 29 \\ 441 &= 29y && \text{divide by } 29 \\ \frac{441}{29} &= y \end{aligned}$$

   A different ratio in the same triangles can be used to obtain

$$\frac{\text{side opposite } \beta}{\text{hypotenuse}} = \frac{20}{29} = \frac{h}{21}$$

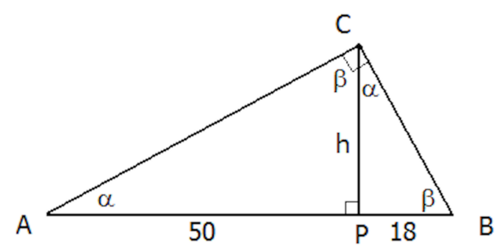
$$\begin{aligned}\frac{20}{29} &= \frac{h}{21} && \text{multiply both sides by } 21\cdot 29\\ 420 &= 29h && \text{divide by 29}\\ \frac{420}{29} &= h\end{aligned}$$

For x, we can simply use the fact that $x+y=29$ and we already computed $y=\dfrac{441}{29}$.

$$\begin{aligned}x+\frac{441}{29} &= 29\\ x &= 29-\frac{441}{29}=\frac{400}{29}\end{aligned}$$

Therefore, $\boxed{x=\dfrac{400}{29},\ y=\dfrac{441}{29},\text{ and } h=\dfrac{420}{29}}$.

6. The picture shows a right triangle. Find the length of h, the height drawn to the hypotenuse.
Solution: Let us first label the points, angles and sides in the triangle. As we proved it in the previous problem, the two new triangles are similar to the original triangle.



Consider now the ratio $\dfrac{\text{side opposite } \beta}{\text{side opposite } \alpha}$ in triangles $\triangle APC$ and $\triangle PBC$. Since these triangles are similar, this ratio is preserved.

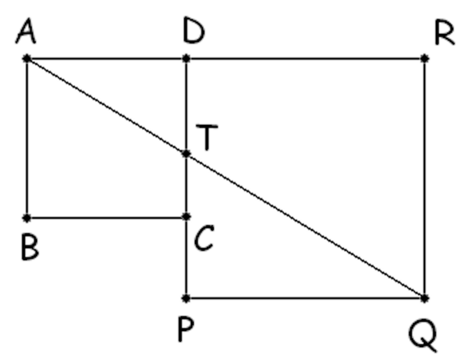
$$\frac{\text{side opposite } \beta}{\text{side opposite } \alpha}=\frac{50}{h}=\frac{h}{18}$$

We solve this equation for h.

$$\begin{aligned}\frac{50}{h} &= \frac{h}{18}\\ 50\cdot 18 &= h^2\\ 900 &= h^2\\ h &= \pm 30\end{aligned}$$

$h=-30$ is ruled out since distances can not be negative. Thus $\boxed{h=30}$.

7. Quadrilaterals $ABCD$ and $PQRD$ are both squares. If $AB=2$ and $PQ=3$, then find the exact value opf the length of TC.



Solution: Triangles ADT and ARQ are similar, because they share and angle at A and they both have a right angle. $AD=2$, $AR=5$, and $RQ=3$.

$$\frac{DT}{AD}=\frac{RQ}{AR} \quad \text{becomes} \quad \frac{x}{2}=\frac{3}{5} \quad \Longrightarrow \quad x=\frac{6}{5}=1.2$$

So $DT=\boxed{1.2\text{ units}}$

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