

1. Simplify each of the following expressions.

$$\begin{array}{ll} \text{a) } \frac{a^2 - 9}{a + 2} \div \left(1 - \frac{5}{a + 2}\right) & \text{c) } \frac{1 - \frac{x^2}{x^2 - 1}}{2 + \frac{3x - 1}{1 - x}} \quad \text{where } |x| \neq 1 \\ \text{b) } \frac{9a - 3b}{9a^2 - b^2} \cdot \frac{15a + 5b}{3} \quad \text{where } |3a| \neq |b| \end{array}$$

2. Find the exact value of each of the following expressions if $x = 2$, $y = \sqrt{3}$, and $z = 0.2009$

$$\begin{array}{ll} \text{a) } \frac{1}{(x - y)(x - z)} + \frac{1}{(z - x)(z - y)} + \frac{1}{(y - x)(y - z)} & \text{d) } \frac{(x + y)^2 - (x - y)^2}{4xy} \\ \text{b) } \frac{1}{x(x + z)} + \frac{1}{z(x + z)} + \frac{1}{x(x - z)} + \frac{1}{z(z - x)} & \text{e) } \frac{(x^2 - y^2 - z^2 - 2yz)(x + y - z)}{(x + y + z)(x^2 + z^2 - 2xz - y^2)} \\ \text{c) } \left(\frac{2x^2 + x}{x^3 - 1} - \frac{x + 1}{x^2 + x + 1}\right) \left(1 + \frac{x + 1}{x} - \frac{x^2 + 5x}{x^2 + x}\right) \end{array}$$

3. Simplify each of the following expressions.

$$\text{a) } \frac{4 - a^2 - 2ab - b^2}{2 + a + b} \quad \text{where } a + b \neq -2 \quad \text{b) } \frac{a^2 + b^2 - c^2 + 2ab}{a^2 - b^2 + c^2 + 2ac} \quad \text{where } |a + c| \neq |b|$$

4. Prove that if $a + b + c = 0$, then $a^3 + a^2c + b^2c - abc + b^3 = 0$

5. Simplify each of the following expressions.

$$\begin{array}{ll} \text{a) } \sqrt{12} + \sqrt{75} - \sqrt{147} & \text{g) } \sqrt{7 - 4\sqrt{3}} - \sqrt{7 + 4\sqrt{3}} \\ \text{b) } \sqrt{28} + \sqrt{7} - \sqrt{63} & \text{h) } \sqrt[3]{7 + 5\sqrt{2}} \\ \text{c) } \sqrt{\sqrt{41} + 4\sqrt{2}} \cdot \sqrt{\sqrt{41} - \sqrt{32}} & \text{i) } \sqrt[3]{20 + 14\sqrt{2}} + \sqrt[3]{20 - 14\sqrt{2}} \\ \text{d) } \sqrt{5\sqrt{3} + \sqrt{59}} \cdot \sqrt{\sqrt{75} - \sqrt{59}} & \text{j) } \sqrt[3]{10 + 6\sqrt{3}} + \sqrt[3]{10 - 6\sqrt{3}} \\ \text{e) } \left(\sqrt{6 + \sqrt{11}} + \sqrt{6 - \sqrt{11}}\right)^2 & \text{k) } \sqrt[4]{7 - 4\sqrt{5}} \\ \text{f) } \sqrt{7 + 2\sqrt{6}} - \sqrt{7 - 2\sqrt{6}} \end{array}$$

6. Simplify each of the following expressions.

$$\text{a) } \frac{3 - \sqrt{5}}{3 + \sqrt{5}} + \frac{3 + \sqrt{5}}{3 - \sqrt{5}} \quad \text{b) } \left(\frac{8}{\sqrt{7} + \sqrt{3}} + \frac{12}{\sqrt{7} - \sqrt{3}}\right) (5\sqrt{7} - \sqrt{3})$$

7. Rationalize the denominator in each of the following expressions.

$$\text{a) } \frac{3}{\sqrt{5} - \sqrt{2}} \quad \text{b) } \frac{a}{\sqrt{a} + \sqrt{b}} \quad \text{where } a, b > 0 \quad \text{c) } \frac{\sqrt{7} - \sqrt{2}}{\sqrt{7} + \sqrt{2}}$$

8. Which one is greater?

$$\begin{array}{ll} \text{a) } 2\sqrt{7} \quad \text{or} \quad \frac{1}{\sqrt{7} - \sqrt{6}} & \text{c) } \frac{7}{5 - 3\sqrt{2}} \quad \text{or} \quad \sqrt{72} \\ \text{b) } \sqrt[4]{4} \quad \text{or} \quad \sqrt[5]{5} & \text{d) } 2\sqrt{3} \quad \text{or} \quad \frac{1}{\sqrt{3} - \sqrt{2}} \end{array}$$

9. For what values of k can we factor out $x + 3$ from the polynomial $2x^2 + x + k$?

10. Find the exact value of the following expression.

$$\frac{\sqrt{\sqrt{5}+2} + \sqrt{\sqrt{5}-2}}{\sqrt{\sqrt{5}+1}} - \sqrt{3-2\sqrt{2}}$$

11. We divided a line segment into two parts so that the ratio between the shorter and longer part is the same as the ratio between the longer part and the entire line segment. If R represents this ratio, find the exact value of the following expression.

$$R\left(R^{(R^2+R^{-1})+R^{-1}}\right) + R^{-1}$$

12. Find the integer part in $(\sqrt{3} + \sqrt{2})^6$.

13. If p , q , and r are solutions of the equation $x^3 - x^2 + x - 2 = 0$, then find the exact value of $p^3 + q^3 + r^3$.

14. Is the number $\sqrt[3]{7+4\sqrt{3}} + \sqrt[3]{7-4\sqrt{3}}$ a solution of the equation $x^3 - 3x - 14 = 0$?

Answers

1. Simplify each of the following expressions.

$$(a) \frac{a^2 - 9}{a + 2} \div \left(1 - \frac{5}{a + 2}\right) = a + 3$$

Solution: We first perform the subtraction and then divide by multiplying by the reciprocal.

$$\begin{aligned} \frac{a^2 - 9}{a + 2} \div \left(1 - \frac{5}{a + 2}\right) &= \frac{a^2 - 9}{a + 2} \div \left(\frac{a + 2}{a + 2} - \frac{5}{a + 2}\right) = \frac{a^2 - 9}{a + 2} \div \frac{a + 2 - 5}{a + 2} = \frac{a^2 - 9}{a + 2} \div \frac{a - 3}{a + 2} \\ &= \frac{a^2 - 9}{a + 2} \cdot \frac{a + 2}{a - 3} = \frac{a^2 - 9}{a - 3} = \frac{(a + 3)(a - 3)}{a - 3} = a + 3 \end{aligned}$$

$$(b) \frac{9a - 3b}{9a^2 - b^2} \cdot \frac{15a + 5b}{3} = 5 \quad \text{where } |3a| \neq |b|$$

Solution:

$$(c) \frac{1 - \frac{x^2}{x^2 - 1}}{2 + \frac{3x - 1}{1 - x}} = \frac{1}{(x + 1)^2} \quad \text{where } |x| \neq 1$$

2. Find the exact value of each of the following expressions if $x = 2$, $y = \sqrt{3}$, and $z = 0.2009$

$$(a) \frac{1}{(x - y)(x - z)} + \frac{1}{(z - x)(z - y)} + \frac{1}{(y - x)(y - z)} = 0$$

Solution:

$$\begin{aligned} &\frac{1}{(x - y)(x - z)} + \frac{1}{(z - x)(z - y)} + \frac{1}{(y - x)(y - z)} = \\ &\frac{y - z}{(x - y)(x - z)(y - z)} + \frac{x - y}{(-1)(x - z)(-1)(y - z)(x - y)} + \frac{x - z}{(-1)(x - y)(y - z)(x - z)} = \\ &\frac{(y - z) + (x - y) - (x - z)}{(x - y)(x - z)(y - z)} = \\ &\frac{y - z + x - y - x + z}{(x - y)(x - z)(y - z)} = 0 \end{aligned}$$

$$(b) \frac{1}{x(x + z)} + \frac{1}{z(x + z)} + \frac{1}{x(x - z)} + \frac{1}{z(z - x)} =$$

Solution:

$$\begin{aligned} &\frac{1}{x(x + z)} + \frac{1}{z(x + z)} + \frac{1}{x(x - z)} + \frac{1}{z(z - x)} = \\ &\frac{z(x - z)}{xz(x + z)(x - z)} + \frac{x(x - z)}{xz(x + z)(x - z)} + \frac{z(x + z)}{xz(x + z)(x - z)} + \frac{x(x + z)}{zx(-1)(x - z)(x + z)} = \\ &\frac{z(x - z) + x(x - z) + z(x + z) - x(x + z)}{xz(x + z)(x - z)} = \\ &\frac{xz - z^2 + x^2 - xz + xz + z^2 - x^2 - xz}{xz(x + z)(x - z)} = 0 \end{aligned}$$

$$\begin{aligned}
(c) \quad & \left(\frac{2x^2 + x}{x^3 - 1} - \frac{x + 1}{x^2 + x + 1} \right) \left(1 + \frac{x + 1}{x} - \frac{x^2 + 5x}{x^2 + x} \right) = \\
& \left(\frac{2x^2 + x}{x^3 - 1} - \frac{x + 1}{x^2 + x + 1} \right) \left(1 + \frac{x + 1}{x} - \frac{x^2 + 5x}{x^2 + x} \right) = \\
& \left(\frac{2x^2 + x}{(x - 1)(x^2 + x + 1)} - \frac{(x - 1)(x + 1)}{(x - 1)(x^2 + x + 1)} \right) \left(1 + \frac{x + 1}{x} - \frac{x(x + 5)}{x(x + 1)} \right) = \\
& \frac{2x^2 + x - (x^2 - 1)}{(x - 1)(x^2 + x + 1)} \left(\frac{x(x + 1)}{x(x + 1)} + \frac{(x + 1)^2}{x(x + 1)} - \frac{x(x + 5)}{x(x + 1)} \right) = \\
& \frac{x^2 + x + 1}{(x - 1)(x^2 + x + 1)} \cdot \frac{x^2 - 2x + 1}{x(x + 1)} = \\
& \frac{1}{(x - 1)} \cdot \frac{(x - 1)^2}{x(x + 1)} = \\
& \frac{(x - 1)}{x(x + 1)} = \frac{1}{2 \cdot 3} = \frac{1}{6}
\end{aligned}$$

$$(d) \quad \frac{(x + y)^2 - (x - y)^2}{4xy} = 1$$

Solution:

$$\begin{aligned}
\frac{(x + y)^2 - (x - y)^2}{4xy} &= \frac{x^2 + 2xy + y^2 - (x^2 - 2xy + y^2)}{4xy} \\
&= \frac{x^2 + 2xy + y^2 - x^2 + 2xy - y^2}{4xy} = \frac{4xy}{4xy} = 1
\end{aligned}$$

$$(e) \quad \frac{(x^2 - y^2 - z^2 - 2yz)(x + y - z)}{(x + y + z)(x^2 + z^2 - 2xz - y^2)} = 1$$

Solution:

$$\begin{aligned}
\frac{(x^2 - y^2 - z^2 - 2yz)(x + y - z)}{(x + y + z)(x^2 + z^2 - 2xz - y^2)} &= \frac{(x^2 - (y + z)^2)(x + y - z)}{(x + y + z)((x - z)^2 - y^2)} \\
&= \frac{(x + y + z)(x - y - z)(x + y - z)}{(x + y + z)(x - z + y)(x - z - y)} \\
&= \frac{(x - y - z)(x + y - z)}{(x - z + y)(x - z - y)} \\
&= \frac{(x - y - z)(x + y - z)}{(x - y - z)(x + y - z)} = 1
\end{aligned}$$

3. Simplify each of the following expressions.

$$(a) \quad \frac{4 - a^2 - 2ab - b^2}{2 + a + b} \quad \text{where } a + b \neq -2$$

Solution:

$$\frac{4 - a^2 - 2ab - b^2}{2 + a + b} = \frac{4 - (a + b)^2}{2 + a + b} = \frac{(2 - (a + b))(2 + a + b)}{2 + a + b} = 2 - a - b$$

$$(b) \quad \frac{a^2 + b^2 - c^2 + 2ab}{a^2 - b^2 + c^2 + 2ac} \quad \text{where } |a + c| \neq |b|$$

Solution:

$$\frac{a^2 + b^2 - c^2 + 2ab}{a^2 - b^2 + c^2 + 2ac} = \frac{(a + b)^2 - c^2}{(a + c)^2 - b^2} = \frac{(a + b + c)(a + b - c)}{(a + c + b)(a + c - b)} = \frac{a + b - c}{a - b + c}$$

4. Prove that if $a + b + c = 0$, then $a^3 + a^2c + b^2c - abc + b^3 = 0$

Solution:

$$\begin{aligned}
 a^3 + a^2c + b^2c - abc + b^3 &= a^3 + b^3 + a^2c + b^2c - abc = \\
 &= (a + b)(a^2 - ab + b^2) + c(a^2 + b^2 - ab) \\
 &= (a + b + c)(a^2 - ab + b^2) \\
 &= 0(a^2 - ab + b^2) = 0
 \end{aligned}$$

5. Simplify each of the following expressions.

(a) $\sqrt{12} + \sqrt{75} - \sqrt{147} = 0$

(b) $\sqrt{28} + \sqrt{7} - \sqrt{63} = 0$

(c) $\sqrt{\sqrt{41} + 4\sqrt{2}} \cdot \sqrt{\sqrt{41} - \sqrt{32}} = 3$

Solution:

$$\begin{aligned}
 \sqrt{\sqrt{41} + 4\sqrt{2}} \cdot \sqrt{\sqrt{41} - \sqrt{32}} &= \sqrt{\sqrt{41} + \sqrt{32}} \cdot \sqrt{\sqrt{41} - \sqrt{32}} = \sqrt{(\sqrt{41} + \sqrt{32})(\sqrt{41} - \sqrt{32})} \\
 &= \sqrt{\left((\sqrt{41})^2 - (\sqrt{32})^2\right)} = \sqrt{41 - 32} = \sqrt{9} = 3
 \end{aligned}$$

(d) $\sqrt{5\sqrt{3} + \sqrt{59}} \cdot \sqrt{\sqrt{75} - \sqrt{59}} = 4$

Solution:

$$\begin{aligned}
 \sqrt{5\sqrt{3} + \sqrt{59}} \cdot \sqrt{\sqrt{75} - \sqrt{59}} &= \sqrt{\sqrt{75} + \sqrt{59}} \cdot \sqrt{\sqrt{75} - \sqrt{59}} = \sqrt{(\sqrt{75} + \sqrt{59})(\sqrt{75} - \sqrt{59})} \\
 &= \sqrt{\left((\sqrt{75})^2 - (\sqrt{59})^2\right)} = \sqrt{75 - 59} = \sqrt{16} = 4
 \end{aligned}$$

(e) $\left(\sqrt{6 + \sqrt{11}} + \sqrt{6 - \sqrt{11}}\right)^2 = 22$

Solution:

$$\begin{aligned}
 \left(\sqrt{6 + \sqrt{11}} + \sqrt{6 - \sqrt{11}}\right)^2 &= \left(\sqrt{6 + \sqrt{11}}\right)^2 + \left(\sqrt{6 - \sqrt{11}}\right)^2 + 2\sqrt{6 + \sqrt{11}}\sqrt{6 - \sqrt{11}} \\
 &= 6 + \sqrt{11} + 6 - \sqrt{11} + 2\sqrt{(6 + \sqrt{11})(6 - \sqrt{11})} \\
 &= 12 + 2\sqrt{36 - 11} = 12 + 2\sqrt{25} = 12 + 2 \cdot 5 = 22
 \end{aligned}$$

(f) $\sqrt{7 + 2\sqrt{6}} - \sqrt{7 - 2\sqrt{6}} = 2$

Solution:

$$\begin{aligned}
 \sqrt{7 + 2\sqrt{6}} - \sqrt{7 - 2\sqrt{6}} &= \sqrt{(\sqrt{6} + 1)^2} - \sqrt{(\sqrt{6} - 1)^2} = \\
 &= \sqrt{6} + 1 - (\sqrt{6} - 1) = \sqrt{6} + 1 - \sqrt{6} + 1 = 2
 \end{aligned}$$

(g) $\sqrt{7 - 4\sqrt{3}} - \sqrt{7 + \sqrt{48}} = -2\sqrt{3}$

Solution:

$$\begin{aligned}
 \sqrt{7 - 4\sqrt{3}} - \sqrt{7 + \sqrt{48}} &= \sqrt{7 - 4\sqrt{3}} - \sqrt{7 + 4\sqrt{3}} \\
 &= \sqrt{(2 - \sqrt{3})^2} - \sqrt{(2 + \sqrt{3})^2} \\
 &= 2 - \sqrt{3} - (2 + \sqrt{3}) = 2 - \sqrt{3} - 2 - \sqrt{3} = -2\sqrt{3}
 \end{aligned}$$

(h) $\sqrt[3]{7+5\sqrt{2}} = \sqrt{2} + 1$

Solution:

$$(\sqrt{2} + 1)^3 = (\sqrt{2})^3 + 3(\sqrt{2})^2 + 3(\sqrt{2}) + 1 = 2\sqrt{2} + 6 + 3\sqrt{2} + 1 = 7 + 5\sqrt{2}$$

$$\sqrt[3]{7+5\sqrt{2}} = \sqrt{2} + 1$$

(i) $\sqrt[3]{20+14\sqrt{2}} + \sqrt[3]{20-14\sqrt{2}} = 4$

Solution: $(2 + \sqrt{2})^3 = 20 + 14\sqrt{2}$ and $(2 - \sqrt{2})^3 = 20 - 14\sqrt{2}$

$$\sqrt[3]{20+14\sqrt{2}} + \sqrt[3]{20-14\sqrt{2}} = 2 + \sqrt{2} + 2 - \sqrt{2} = 4$$

(j) $\sqrt[3]{10+6\sqrt{3}} + \sqrt[3]{10-6\sqrt{3}} = 2$

Solution: $(1 + \sqrt{3})^3 = 10 + 6\sqrt{3}$ and $(1 - \sqrt{3})^3 = 10 - 6\sqrt{3}$

$$\sqrt[3]{10+6\sqrt{3}} + \sqrt[3]{10-6\sqrt{3}} = 1 + \sqrt{3} + 1 - \sqrt{3} = 2$$

(k) $\sqrt[4]{7-4\sqrt{5}} = \text{undefined}$

$$\sqrt{5} > 2$$

$$-4\sqrt{5} < -8$$

$$7 - 4\sqrt{5} < -1$$

6. Simplify each of the following expressions.

(a) $\frac{3-\sqrt{5}}{3+\sqrt{5}} + \frac{3+\sqrt{5}}{3-\sqrt{5}} = 7$

Solution:

$$\frac{3-\sqrt{5}}{3+\sqrt{5}} + \frac{3+\sqrt{5}}{3-\sqrt{5}} = \frac{(3-\sqrt{5})^2 + (3+\sqrt{5})^2}{(3+\sqrt{5})(3-\sqrt{5})} = \frac{14-6\sqrt{5}+14+6\sqrt{5}}{9-5} = \frac{28}{4} = 7$$

(b) $\left(\frac{8}{\sqrt{7}+\sqrt{3}} + \frac{12}{\sqrt{7}-\sqrt{3}}\right)(5\sqrt{7}-\sqrt{3}) = 172$

Solution:

$$\begin{aligned} & \left(\frac{8}{\sqrt{7}+\sqrt{3}} + \frac{12}{\sqrt{7}-\sqrt{3}}\right)(5\sqrt{7}-\sqrt{3}) = \\ &= \frac{8(\sqrt{7}-\sqrt{3}) + 12(\sqrt{7}+\sqrt{3})}{(\sqrt{7}+\sqrt{3})(\sqrt{7}-\sqrt{3})}(5\sqrt{7}-\sqrt{3}) \\ &= \frac{8\sqrt{7}-8\sqrt{3}+12\sqrt{7}+12\sqrt{3}}{7-3}(5\sqrt{7}-\sqrt{3}) \\ &= \frac{20\sqrt{7}+4\sqrt{3}}{4}(5\sqrt{7}-\sqrt{3}) = (5\sqrt{7}+\sqrt{3})(5\sqrt{7}-\sqrt{3}) \\ &= 25(7)-3 = 175-3 = 172 \end{aligned}$$

7. Rationalize the denominator in each of the following expressions.

$$(a) \frac{3}{\sqrt{5} - \sqrt{2}} = \sqrt{5} + \sqrt{2}$$

$$(b) \frac{a}{\sqrt{a} + \sqrt{b}} = \frac{a(\sqrt{a} - \sqrt{b})}{a - b} \quad \text{where } a, b > 0$$

$$(c) \frac{\sqrt{7} - \sqrt{2}}{\sqrt{7} + \sqrt{2}} = \frac{9 - 2\sqrt{14}}{5}$$

8. Which one is greater?

$$(a) 2\sqrt{7} \quad \text{or} \quad \frac{1}{\sqrt{7} - \sqrt{6}} \quad 2\sqrt{7}$$

$$(b) \sqrt[4]{4} \quad \text{or} \quad \sqrt[5]{5} \quad \sqrt[4]{4}$$

$$(c) \frac{7}{5 - 3\sqrt{2}} \quad \text{or} \quad \sqrt{72} \quad \frac{7}{5 - 3\sqrt{2}}$$

Solution:

$$\frac{7}{5 - 3\sqrt{2}} = \frac{7(5 + 3\sqrt{2})}{(5 - 3\sqrt{2})(5 + 3\sqrt{2})} = \frac{7(5 + 3\sqrt{2})}{5^2 - (3\sqrt{2})^2} = \frac{7(5 + 3\sqrt{2})}{25 - 18} = 5 + 3\sqrt{2}$$

$$\sqrt{72} = 6\sqrt{2}$$

$$5 + 3\sqrt{2} > 6\sqrt{2}$$

$$5 > 3\sqrt{2}$$

$$\sqrt{25} > \sqrt{18}$$

$$(d) 2\sqrt{3} \quad \text{or} \quad \frac{1}{\sqrt{3} - \sqrt{2}} \quad 2\sqrt{3}$$

9. For what values of k can we factor out $x + 3$ from the polynomial $2x^2 + x + k$? -15

10. Find the exact value of the following expression.

$$\begin{aligned} & \frac{\sqrt{\sqrt{5} + 2} + \sqrt{\sqrt{5} - 2}}{\sqrt{\sqrt{5} + 1}} - \sqrt{3 - 2\sqrt{2}} = \\ &= \frac{\sqrt{(\sqrt{\sqrt{5} + 2} + \sqrt{\sqrt{5} - 2})^2}}{\sqrt{\sqrt{5} + 1}} - \sqrt{(\sqrt{2} - 1)^2} \\ &= \frac{\sqrt{(\sqrt{5} + 2) + (\sqrt{5} - 2) + 2\sqrt{\sqrt{5} + 2}\sqrt{\sqrt{5} - 2}}}{\sqrt{\sqrt{5} + 1}} - (\sqrt{2} - 1) \\ &= \frac{\sqrt{2\sqrt{5} + 2\sqrt{(\sqrt{5} + 2)(\sqrt{5} - 2)}}}{\sqrt{\sqrt{5} + 1}} - (\sqrt{2} - 1) \\ &= \frac{\sqrt{2\sqrt{5} + 2\sqrt{1}}}{\sqrt{\sqrt{5} + 1}} - (\sqrt{2} - 1) = \frac{\sqrt{2\sqrt{5} + 2}}{\sqrt{\sqrt{5} + 1}} - (\sqrt{2} - 1) \\ &= \frac{\sqrt{2}\sqrt{\sqrt{5} + 1}}{\sqrt{\sqrt{5} + 1}} - (\sqrt{2} - 1) = \sqrt{2} - (\sqrt{2} - 1) = 1 \end{aligned}$$

11. We divided a line segment into two parts so that the ratio between the shorter and longer part is the same as the ratio between the longer part and the entire line segment. If R represents this ratio, find the exact value of the following expression.

$$R \left(R^{(R^2 + R^{-1}) + R^{-1}} \right) + R^{-1}$$

2

12. Find the integer part in $(\sqrt{3} + \sqrt{2})^6$. 969

Solution:

$$\begin{aligned} (\sqrt{3} + \sqrt{2})^6 &= \left((\sqrt{3} + \sqrt{2})^2 \right)^3 = (5 + 2\sqrt{6})^3 \\ &= (5 + \sqrt{24})^3 = 125 + 3(25)\sqrt{24} + 3(5)(24) + 24\sqrt{24} \\ &= 125 + 75\sqrt{24} + 360 + 24\sqrt{24} = 485 + 99\sqrt{24} \\ &= 485 + \sqrt{235224} \quad \sqrt{235224} = 484.99896 \\ &485 + 484 = 969 \end{aligned}$$

13. If p , q , and r are solutions of the equation $x^3 - x^2 + x - 2 = 0$, then find the exact value of $p^3 + q^3 + r^3$.

$$\begin{aligned} x^3 - x^2 + x - 2 &= (x - p)(x - q)(x - r) \\ x^3 - x^2 + x - 2 &= x^3 + x^2(-p - q - r) + x(pq + pr + qr) - pqr \\ -1 &= -p - q - r \\ 1 &= pq + pr + qr \\ -2 &= -pqr \end{aligned}$$

$$\begin{aligned} 1 &= p + q + r \\ 1 &= pq + pr + qr \\ 2 &= pqr \end{aligned}$$

Let us first find the exact value of $p^2 + q^2 + r^2$. Since $(p + q + r)^2 = 2pq + 2pr + 2qr + p^2 + q^2 + r^2$, we have that

$$p^2 + q^2 + r^2 = (p + q + r)^2 - 2(pq + pr + qr) = 1^2 - 2(1) = -1$$

which shows that not all three of p , q , and r can be real. We start with $(p + q + r)^3$.

$$(p + q + r)^3 = p^3 + q^3 + r^3 + 3pq^2 + 3p^2q + 3pr^2 + 3p^2r + 3qr^2 + 3q^2r + 6pqr$$

Let us introduce a new variable $A = pq^2 + p^2q + pr^2 + p^2r + qr^2 + q^2r$. With this notation,

$$\begin{aligned} p^3 + q^3 + r^3 &= (p + q + r)^3 - 3A - 6pqr \\ &= 1^3 - 3A - 6(2) \\ &= -11 - 3A \end{aligned}$$

Let us now similarly compute $(p + q + r)(p^2 + q^2 + r^2)$.

$$(p + q + r)(p^2 + q^2 + r^2) = p^3 + q^3 + r^3 + pq^2 + p^2q + pr^2 + p^2r + qr^2 + q^2r$$

and so

$$\begin{aligned} p^3 + q^3 + r^3 &= (p + q + r)(p^2 + q^2 + r^2) - (pq^2 + p^2q + pr^2 + p^2r + qr^2 + q^2r) \\ &= 1 \cdot (-1) - A = -1 - A \end{aligned}$$

If we denote $p^3 + q^3 + r^3$ by X , we have a simple system of linear equations we can solve

$$\begin{aligned} X &= -11 - 3A \\ X &= -1 - A \end{aligned}$$

and obtain that $A = -5$ and $X = 4$.

14. Is the number $\sqrt[3]{7+4\sqrt{3}} + \sqrt[3]{7-4\sqrt{3}}$ a solution of the equation $x^3 - 3x - 14 = 0$?

Solution: Let us denote $\sqrt[3]{7+4\sqrt{3}} + \sqrt[3]{7-4\sqrt{3}}$ by x and compute x^3 first.

$$\begin{aligned} & \left(\sqrt[3]{7+4\sqrt{3}} + \sqrt[3]{7-4\sqrt{3}} \right)^3 = \\ &= \left(\sqrt[3]{7+4\sqrt{3}} \right)^3 + 3 \left(\sqrt[3]{7+4\sqrt{3}} \right)^2 \left(\sqrt[3]{7-4\sqrt{3}} \right) + \\ & \quad + 3 \left(\sqrt[3]{7+4\sqrt{3}} \right) \left(\sqrt[3]{7-4\sqrt{3}} \right)^2 + \left(\sqrt[3]{7-4\sqrt{3}} \right)^3 \\ &= 7 + 4\sqrt{3} + 3\sqrt[3]{(7+4\sqrt{3})^2(7-4\sqrt{3})} + 3\sqrt[3]{(7+4\sqrt{3})(7-4\sqrt{3})^2} + 7 - 4\sqrt{3} \\ &= 14 + 3 \left(\sqrt[3]{(7+4\sqrt{3})^2(7-4\sqrt{3})} + \sqrt[3]{(7+4\sqrt{3})(7-4\sqrt{3})^2} \right) \\ &= 14 + 3\sqrt[3]{(7+4\sqrt{3}) \left(\sqrt[3]{7-4\sqrt{3}} \right) \left(\sqrt[3]{7+4\sqrt{3}} + \sqrt[3]{7-4\sqrt{3}} \right)} \\ &= 14 + 3\sqrt[3]{49-48} \left(\sqrt[3]{7+4\sqrt{3}} + \sqrt[3]{7-4\sqrt{3}} \right) \\ &= 14 + 3 \cdot 1 \cdot x = 14 + 3x \end{aligned}$$

Since we have that $x^3 = 3x + 14$, it is indeed the solution of the equation $x^3 - 3x - 14 = 0$.