

Sample Problems

1. Find an equation for the circle centered at $(2, -1)$ with radius $r = 5$ units.
2. Graph the equation $10x + x^2 + y^2 = 6(y - 5)$.
3. Find the coordinates of all points where the circle $x^2 + (y - 3)^2 = 50$ and the line $y = -2x + 8$ intersect each other.
4. Consider the circle $x^2 + y^2 = 10$. Find an equation for the tangent line drawn to the circle at the point $(3, 1)$.
5. Find the points where the circles $(x + 2)^2 + (y + 2)^2 = 50$ and $(x - 2)^2 + (y - 1)^2 = 25$ intersect each other.
6. Consider the circles $x^2 + y^2 = 36$ and $(x - 5)^2 + y^2 = 16$. Let C be the intersection of the two common tangent lines drawn to the circle.
 - a) Find the coordinates of C .
 - b) Consider one of the common tangent lines. Find the distance between the points of tangency.
7. Given are the points $A(-3, 0)$ and $B(0, -4)$. Consider the circle $x^2 + y^2 - 16x - 4y + 43 = 0$. Find the tangent lines that are parallel with line segment AB .

Practice Problems

1. Find an equation for each of the following circles. C will denote the center and r the radius.
 - a) $C(6, 0)$ $r = 7$
 - b) $C(0, -2)$ $r = 2$
 - c) $C(-8, 3)$ $r = \sqrt{3}$
2. Graph the equation $x^2 + y^2 + 14y = 4(x + y - 1)$
3. Find the coordinates of all point(s) where the circle and the line intersect each other.
 - a) $(x + 5)^2 + (y - 7)^2 = 8$ and $x + y = 6$
 - c) $x^2 + (y + 4)^2 = 25$ and $y = x - 24$
 - b) $(x - 2)^2 + (y + 1)^2 = 50$ and $y = -x + 9$
 - d) $(x - 3)^2 + (y + 1)^2 = 25$ and $y = -x + 1$
4. Given the equation of a circle and a point P on it, find an equation for the tangent line drawn to the circle at the point P .
 - a) $x^2 + y^2 = 100$ and $P(-8, 6)$
 - b) $(x - 3)^2 + (y + 3)^2 = 50$ and $P(10, -4)$
5. Find the coordinates of all points where the given circles intersect each other.
 - a) $(x - 1)^2 + (y - 2)^2 = 10$ and $(x - 10)^2 + (y - 5)^2 = 40$
 - b) $(x - 3)^2 + y^2 = 20$ and $x^2 + (y - 1)^2 = 50$
 - c) $x^2 + (y - 8)^2 = 26$ and $(x - 7)^2 + (y - 3)^2 = 4$
6. Consider the circles $(x + 5)^2 + y^2 = 100$ and $x^2 + y^2 = 49$. Let C be the intersection of the two common tangent lines drawn to the circle.
 - a) Find the coordinates of C .
 - b) Consider one of the common tangent lines. Find the distance between the points of tangency.

Sample Problems - Answers

- 1.) $(x-2)^2 + (y+1)^2 = 25$ 2.) $(x+5)^2 + (y-3)^2 = 4$ 3.) $(-1, 10)$ and $(5, -2)$
4.) $y = -3x + 10$ 5.) $(-1, 5)$ and $(5, -3)$ 6.) a) $(15, 0)$ b) $\sqrt{21}$
7.) $y = -\frac{4}{3}x + \frac{13}{3}$ and $y = -\frac{4}{3}x + 21$

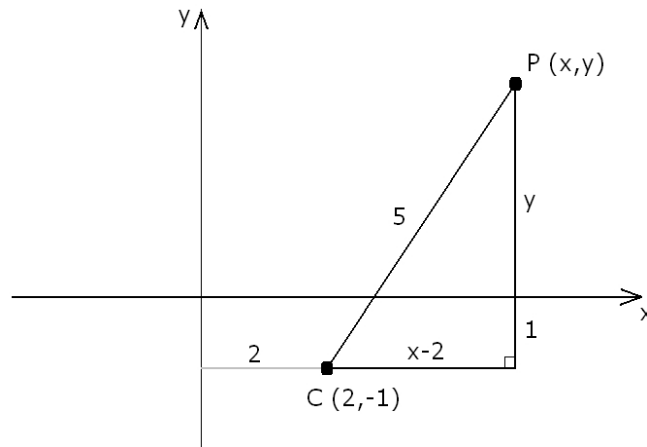
Practice Problems - Answers

- 1.) a) $(x-6)^2 + y^2 = 49$ b) $x^2 + (y+2)^2 = 4$ c) $(x+8)^2 + (y-3)^2 = 3$
2.) $(x-2)^2 + (y+5)^2 = 25$ center: $(2, -5)$ radius: 5
3.) a) $(-3, 9)$ b) $(3, 6)$ and $(9, 0)$ c) they don't intersect d) $(-1, 2)$ and $(6, -5)$
4.) $\frac{4}{3}(x+8) = y-6$ or $y = \frac{4}{3}x + \frac{50}{3}$ b) $7(x-10) = y+4$ or $y = 7x - 74$
5.) a) $(4, 3)$ b) $(7, 2)$ and $(5, -4)$ c) the circles do not intersect
6.) a) $\left(\frac{35}{3}, 0\right)$ b) 4

Sample Problems - Solutions

1. Find an equation for the circle centered at $(2, -1)$ with radius $r = 5$ units.

Solution: Let $P(x, y)$ be a general point on the circle. If the point P is on the circle, then it must be 5 units away from the point $C(2, -1)$. Consider the right triangle shown on the picture below. The sides of this triangle are $|x - 2|$, $|y + 1|$, and 5. The Pythagorean theorem, stated for this triangle is $(x - 2)^2 + (y + 1)^2 = 25$, which is the equation for the circle.

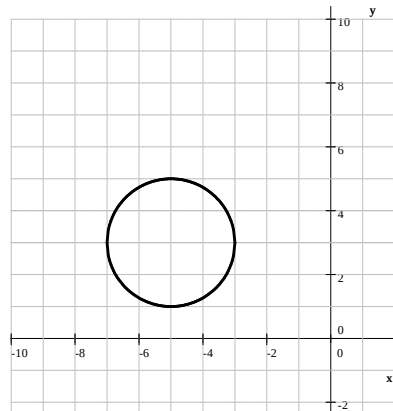


2. Graph the equation $10x + x^2 + y^2 = 6(y - 5)$

Solution: Complete the squares to read the center and radius.

$$\begin{aligned}
 10x + x^2 + y^2 &= 6(y - 5) \\
 10x + x^2 + y^2 &= 6y - 30 \\
 x^2 + 10x + y^2 - 6y &= -30 \\
 \underbrace{x^2 + 10x + 25}_{(x+5)^2} - 25 + \underbrace{y^2 - 6y + 9}_{(y-3)^2} - 9 &= -30 \\
 (x + 5)^2 + (y - 3)^2 - 34 &= -30 \\
 (x + 5)^2 + (y - 3)^2 &= 4
 \end{aligned}$$

Thus the center is $(-5, 3)$ and the radius 2.



3. Find the coordinates of all points where the circle $x^2 + (y - 3)^2 = 50$ and the line $y = -2x + 8$ intersect each other.

Solution: We need to solve the following system of equations:

$$\begin{cases} x^2 + (y - 3)^2 = 50 \\ y = -2x + 8 \end{cases}$$

We will use substitution. We substitute $y = -2x + 8$ in the first equation and solve for x .

$$\begin{array}{rclcl} x^2 + (-2x + 8 - 3)^2 & = & 50 & & 5x^2 - 20x - 25 = 0 \\ x^2 + (-2x + 5)^2 & = & 50 & & 5(x^2 - 4x - 5) = 0 \\ x^2 + 4x^2 - 20x + 25 & = & 50 & & 5(x - 5)(x + 1) = 0 \\ 5x^2 - 20x + 25 & = & 50 & & x_1 = 5 \quad x_2 = -1 \end{array}$$

We now use the second equation to find the y -value belonging to the x -values we just obtained. If $x = 5$, then $y = -2(5) + 8 = -2$. If $x = -1$, then $y = -2(-1) + 8 = 10$. Thus the two points are $(5, -2)$ and $(-1, 10)$. We check: both points should be on both the circle and the line.

4. Consider the circle $x^2 + y^2 = 10$. Find an equation for the tangent line drawn to the circle at the point $(3, 1)$.

Solution: The tangent line is perpendicular to the radius drawn to the point of tangency. We can easily find the slope of this radius: the slope of the line segment connecting $(0, 0)$ and $(3, 1)$ is $\frac{1}{3}$ since

$$\text{slope} = m = \frac{\text{rise}}{\text{run}} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{1 - 0}{3 - 0} = \frac{1}{3}$$

The tangent line then must have slope -3 , the negative reciprocal of $\frac{1}{3}$. It must also pass through the point $(3, 1)$. The point-slope form of this line's equation is then $y - 1 = -3(x - 3)$. We simplify this and obtain $y = -3x + 10$.

5. Find the points where the circles $(x + 2)^2 + (y + 2)^2 = 50$ and $(x - 2)^2 + (y - 1)^2 = 25$ intersect each other.

Solution: We need to solve the following system:

$$\begin{cases} (x + 2)^2 + (y + 2)^2 = 50 \\ (x - 2)^2 + (y - 1)^2 = 25 \end{cases}$$

We first multiply out the complete squares, combine like terms, and reduce the right-hand sides to zero. The first equation:

$$\begin{array}{rcl} x^2 + 4x + 4 + y^2 + 4y + 4 - 50 & = & 0 \\ x^2 + 4x + y^2 + 4y - 42 & = & 0 \end{array}$$

Similarly, the second equation

$$\begin{array}{rcl} x^2 - 4x + 4 + y^2 - 2y + 1 - 25 & = & 0 \\ x^2 - 4x + y^2 - 2y - 20 & = & 0 \end{array}$$

We will multiply the second equation by -1 and add the two equations. This will cancel out all quadratic terms.

$$\begin{aligned}x^2 + 4x + y^2 + 4y - 42 &= 0 \\ -x^2 + 4x - y^2 + 2y + 20 &= 0\end{aligned}$$

The sum of the two equations is

$$\begin{aligned}8x + 6y - 22 &= 0 && \text{divide both sides by 2} \\ 4x + 3y - 11 &= 0\end{aligned}$$

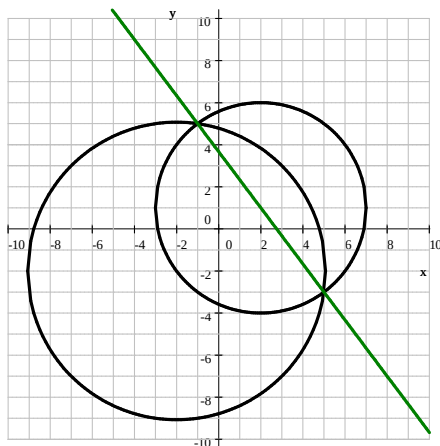
We now solve for y

$$y = \frac{-4x + 11}{3} = -\frac{4}{3}x + \frac{11}{3}$$

We substitute this into the first equation and solve for x

$$\begin{aligned}(x+2)^2 + (y+2)^2 &= 50 && \text{and } y = -\frac{4}{3}x + \frac{11}{3} \\ (x+2)^2 + \left(\underbrace{-\frac{4}{3}x + \frac{11}{3}}_y + 2\right)^2 &= 50 && \frac{11}{3} + 2 = \frac{17}{3} \\ (x+2)^2 + \left(\frac{-4x+17}{3}\right)^2 &= 50 \\ (x+2)^2 + \frac{(-4x+17)^2}{3^2} &= 50 && \text{since } \left(\frac{a}{b}\right)^2 = \frac{a^2}{b^2} \\ (x+2)^2 + \frac{(-4x+17)^2}{9} &= 50 && \text{multiply by 9} \\ 9(x+2)^2 + (-4x+17)^2 &= 450 \\ 9(x^2 + 4x + 4) + 16x^2 - 136x + 289 &= 450 \\ 9x^2 + 36x + 36 + 16x^2 - 136x + 289 &= 450 \\ 25x^2 - 100x - 125 &= 0 \\ 25(x^2 - 4x - 5) &= 0 \\ 25(x-5)(x+1) &= 0 && \implies x_1 = 5 \quad x_2 = -1\end{aligned}$$

We now find the y values belonging to the x -values, using $y = -\frac{4}{3}x + \frac{11}{3}$ or better yet, $y = \frac{-4x+11}{3}$. If $x = 5$, then $y = \frac{-4(5)+11}{3} = -3$ and if $x = -1$, then $y = \frac{-4(-1)+11}{3} = 5$. Thus the two circles intersect at the points $(-1, 5)$ and $(5, -3)$.

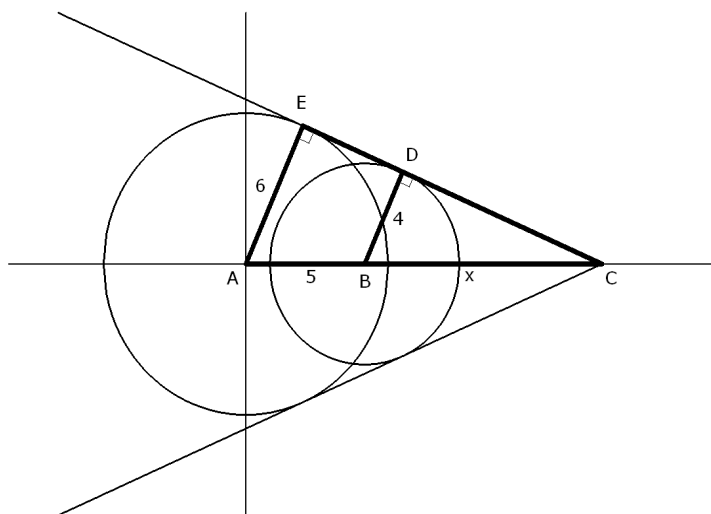


6. Consider the circles $x^2 + y^2 = 36$ and $(x - 5)^2 + y^2 = 16$. Let C be the intersection of the two common tangent lines drawn to the circle.

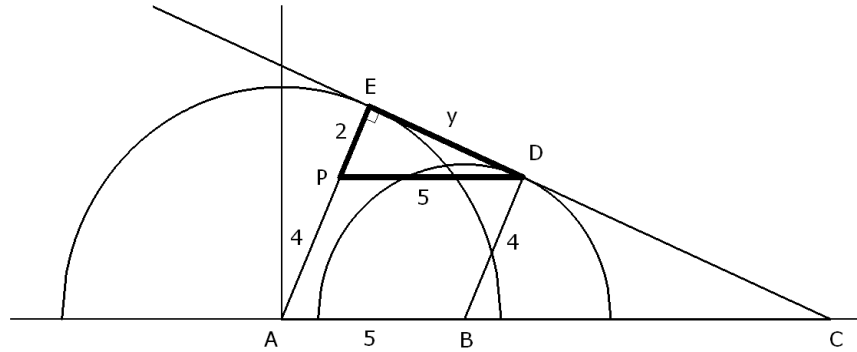
a) Find the coordinates of C .

Solution: Using the notation shown on the picture below, triangles ACE and BCD are similar.

Based on that, we write the equation $\frac{6}{4} = \frac{x+5}{x}$. The solution of this equation is 10. Thus C is $(15, 0)$.



b) Consider one of the common tangent lines. Find the distance between the point of tangency. Solution: We draw a horizontal line through the point D . Let us denote the intersection of AE and the horizontal line by P . The quadrilateral $ABDP$ is a parallelogram since opposite sides are parallel. Consequently, opposite sides are equally long. This means that $AP = 4$ and so $PE = 2$. Also, $PD = 5$. We can now compute y via the Pythagorean theorem: $y = \sqrt{21}$.



7. Given are the points $A(-3, 0)$ and $B(0, -4)$. Consider the circle $x^2 + y^2 - 16x - 4y + 43 = 0$. Find the tangent lines that are parallel with line segment AB .

Solution: The circle is

$$\begin{aligned} x^2 + y^2 - 16x - 4y + 43 &= 0 \\ x^2 - 16x + y^2 - 4y + 43 &= 0 \\ x^2 - 16x + 64 - 64 + y^2 - 4y + 4 - 4 + 43 &= 0 \\ (x - 8)^2 + (y - 2)^2 &= 25 \end{aligned}$$

Thus the center is $(8, 2)$ and the radius is 5. The slope of the line segment AB :

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-4 - 0}{0 - (-3)} = -\frac{4}{3}$$

The tangent line and the radius drawn to the point of tangency are perpendicular: thus we want a straight line that is perpendicular to the line segment AB and passes through the center. This means a slope of $\frac{3}{4}$, and passing through $(8, 2)$. The point-slope form of this line is $y - 2 = \frac{3}{4}(x - 8)$. We intersect this with the circle to find the points of tangency, i.e. solve the system

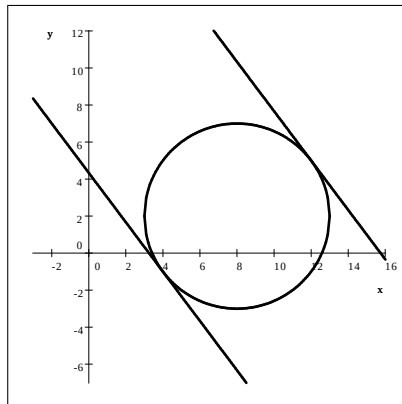
$$\begin{cases} (x - 8)^2 + (y - 2)^2 = 25 \\ y - 2 = \frac{3}{4}(x - 8) \end{cases}$$

$$\begin{aligned}
(x-8)^2 + \left(\frac{3}{4}(x-8)\right)^2 &= 25 \\
(x-8)^2 + \frac{9}{16}(x-8)^2 &= 25 & 1 + \frac{9}{16} = \frac{25}{16} \\
\frac{25}{16}(x-8)^2 &= 25 & \text{divide by 25} \\
\frac{1}{16}(x-8)^2 &= 1 & \text{multiply by 16} \\
(x-8)^2 &= 16 & \text{subtract 16} \\
(x-8)^2 - 16 &= 0 \\
(x-8+4)(x-8-4) &= 0 \\
(x-4)(x-12) &= 0 \implies x_1 = 4 & x_2 = 12
\end{aligned}$$

We find the y -values: when $x = 4$, then $y - 2 = \frac{3}{4}(4 - 8) \implies y_1 = -1$, and when $x = 12$, then $y - 2 = \frac{3}{4}(12 - 8) \implies y_2 = 5$. Thus the points of tangency are $(4, -1)$ and $(12, 5)$. The slope of both tangent lines must be the same as that of line segment AB . Thus the tangent lines:

$$\begin{aligned}
m &= -\frac{4}{3} & T_1 = (4, -1) & \implies y + 1 = -\frac{4}{3}(x - 4) \\
m &= -\frac{4}{3} & T_2 = (12, 5) & \implies y - 5 = -\frac{4}{3}(x - 12)
\end{aligned}$$

After simplifying these equations we get $y = -\frac{4}{3}x + \frac{13}{3}$ and $y = -\frac{4}{3}x + 21$



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