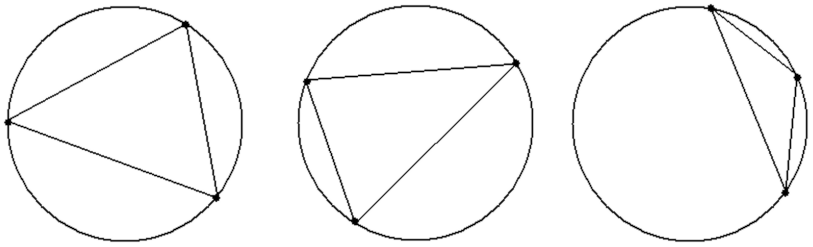
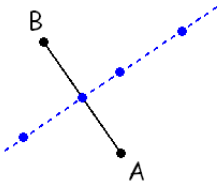
Theorem: For every triangle, there exists a circle that passes through all three vertices of the triangle. We call such a circle the **circumscribed circle**.



We will prove this theorem.

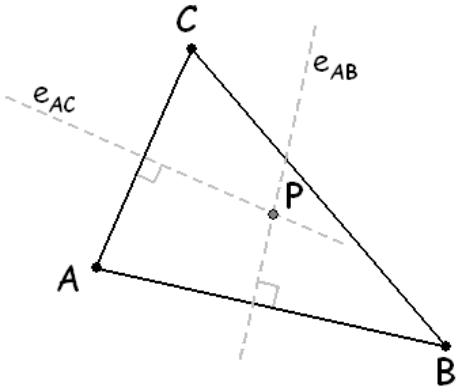
Definition: Suppose that AB is a line segment in a plane. The **perpendicular bisector** of line segment AB is the set of all points in the plane, equidistant to A and B.

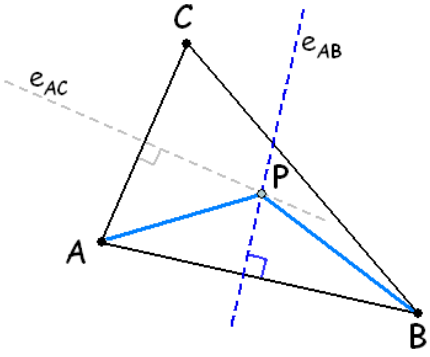


This might seem like a strange definition. Why couldn't we define it as a line perpendicular to the line segment and passing through its midpoint? While these properties are true and useful, geometric objects are often defined in terms of distances.

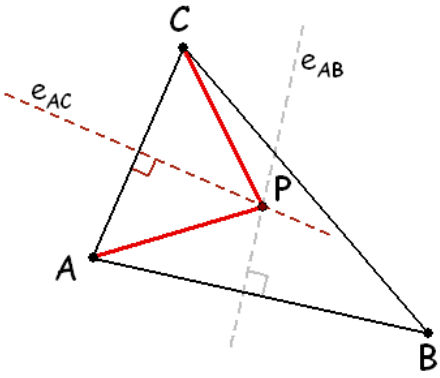
Notation: Given two geometric objects A and B, we will denote the distance between them by $d(A, B)$.

Proof: Suppose that ABC is any triangle. Let e_{AB} be the perpendicular bisector of line segment AB, and let e_{AC} be the perpendicular bisector of line segment AC. Let P be the point at which the two bisectors intersect each other.





Point P is on the perpendicular bisector e_{AB}. Therefore, its distances from A and B are the same. Using notation: $d(P, A) = d(P, B)$.



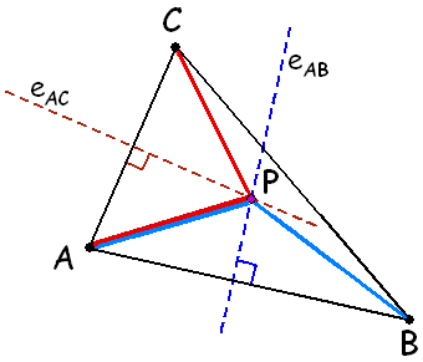
Point P is on the perpendicular bisector e_{AC}. Therefore, its distances from A and C are the same. Using notation: $d(P, A) = d(P, C)$.

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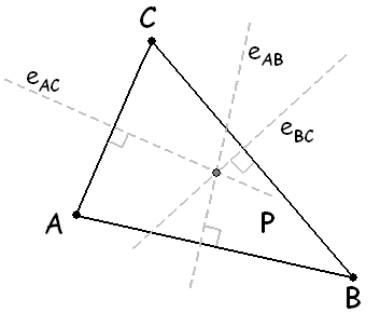
This means that point P is the same distance from A, B, and C:

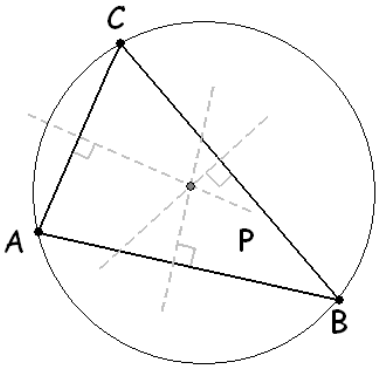
$$d(P,B) = d(P,A) = d(P,C)$$

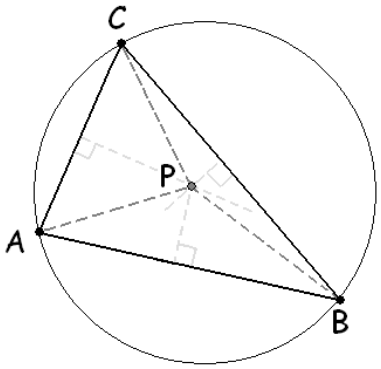
There are two immediate consequences of this.



First, if we were to draw the third perpendicular bisector, e_{BC}, that too will pass through point P. Second, if we draw a circle centered at P and with radius $d(P,A)$, this circle will contain all three points A, B, and C. This concludes our proof. ■







Theorem: Given any triangle ABC, the perpendicular bisectors of the three sides intersect each other in a single point P. This point is the center of the circle that contains all three points A, B, and C. This circle is called **the circumscribed circle** of the triangle. Unless specifically stated otherwise, R will denote the radius of the circumscribed circle.

Exercise: Prove that if the center of the circumscribed circle is on the triangle, then the triangle has a right angle.

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