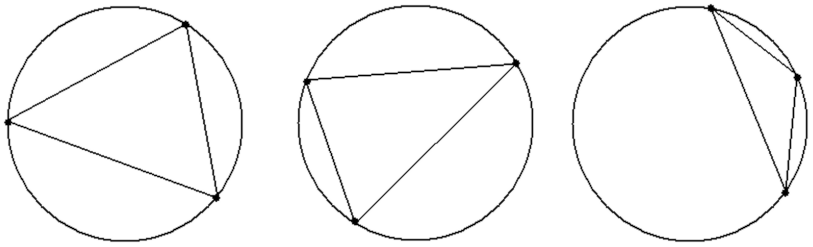
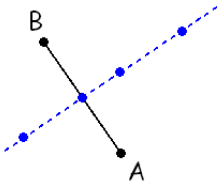
Theorem: For every triangle, there exists a circle that passes through all three vertices of the triangle. We call such a circle the **circumscribed circle**.



We will prove this theorem.

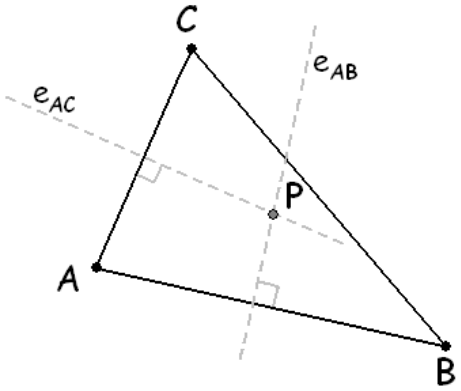
Definition: Suppose that AB is a line segment in a plane. The **perpendicular bisector** of line segment AB is the set of all points in the plane, equidistant to A and B.

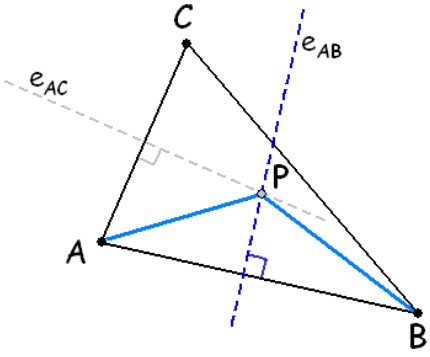


This might seem like a strange definition. Why couldn't we define it as a line perpendicular to the line segment and passing through its midpoint? While these properties are true and useful, geometric objects are often defined in terms of distances.

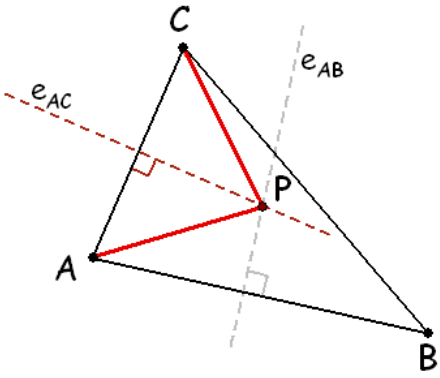
Notation: Given two geometric objects A and B, we will denote the distance between them by $d(A, B)$.

Proof: Suppose that ABC is any triangle. Let e_{AB} be the perpendicular bisector of line segment AB, and let e_{AC} be the perpendicular bisector of line segment AC. Let P be the point at which the two bisectors intersect each other.





Point P is on the perpendicular bisector e_{AB}. Therefore, its distances from A and B are the same. Using notation: $d(P, A) = d(P, B)$.



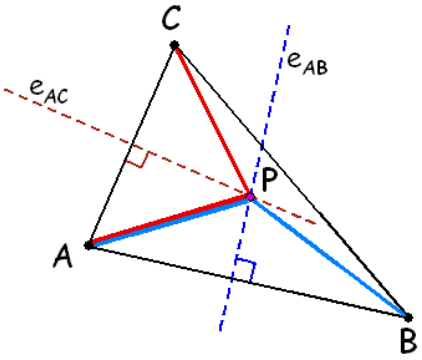
Point P is on the perpendicular bisector e_{AC}. Therefore, its distances from A and C are the same. Using notation: $d(P, A) = d(P, C)$.

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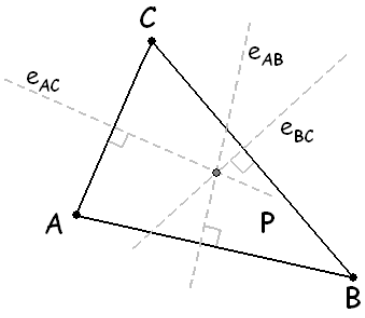
This means that point P is the same distance from A, B, and C:

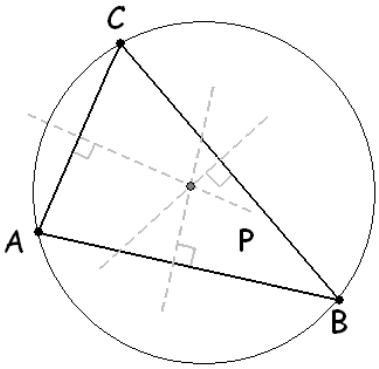
$$d(P,B) = d(P,A) = d(P,C)$$

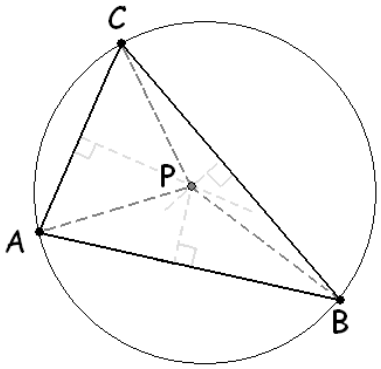
There are two immediate consequences of this.



First, if we were to draw the third perpendicular bisector, e_{BC}, that too will pass through point P. Second, if we draw a circle centered at P and with radius $d(P,A)$, this circle will contain all three points A, B, and C. This concludes our proof. ■

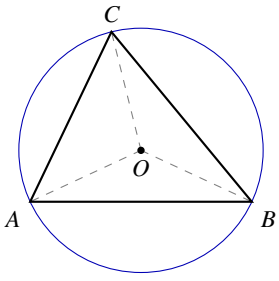




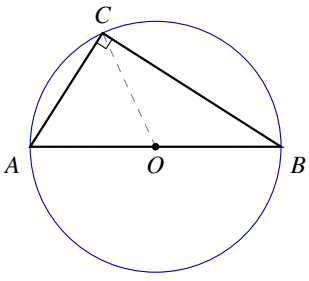


Definition: The **circumscribed circle** of a triangle is the circle that passes through all three vertices of the triangle. Its center is the point where the perpendicular bisectors of the three sides intersect each other. From now on, we call the center O. Unless specifically stated otherwise, R will denote the radius of the circumscribed circle.

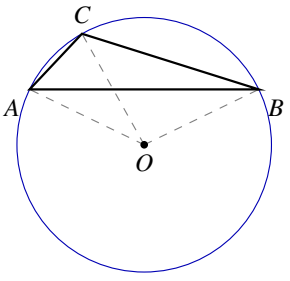
Where the center is depends on the largest angle of the triangle.



If all angles are acute, the center is inside the triangle.



If the triangle has a right angle, the center is the midpoint of the hypotenuse.

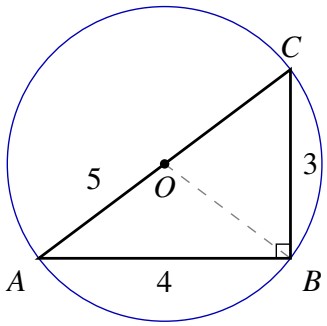


If the triangle has an obtuse angle, the center is outside the triangle.

Now that we know the Pythagorean theorem, we can compute the radius of the circumscribed circle in some cases. The easiest case is the right triangle.

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Example 1. The legs of a right triangle are 3 and 4 units long. Find the radius of the circle circumscribed around the triangle.



Solution: First we find the hypotenuse c. By the Pythagorean theorem,

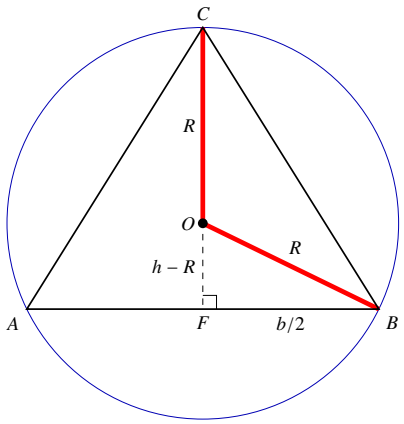
$$c^2 = 3^2 + 4^2 = 9 + 16 = 25$$

This equation has two solutions, $c = 5$ and $c = -5$. Since c is a length, it can't be negative, so $c = 5$.

The center of the circumscribed circle is the midpoint of the hypotenuse, so the hypotenuse is a diameter of the circle. The radius is half of it: $R = \dfrac{5}{2} = 2.5$ units.

Why is the center the midpoint of the hypotenuse? A right triangle is half of a rectangle, cut along a diagonal. The two diagonals of a rectangle are equally long, and they cut each other in half. So the midpoint of the diagonal is equally far from all four corners of the rectangle, and in particular from the three vertices of the triangle. (This fact is also known as Thales's theorem.)

Suppose that triangle ABC is isosceles, with $AC = BC$. Let F be the midpoint of the base AB. Point C is equally far from A and B, so C is on the perpendicular bisector of AB. This means that the perpendicular bisector of the base is the line CF, and so CF is also the height belonging to the base.



The center O of the circumscribed circle is on every perpendicular bisector, so O is on the height CF. This gives us a right triangle, OFB, with the right angle at F:

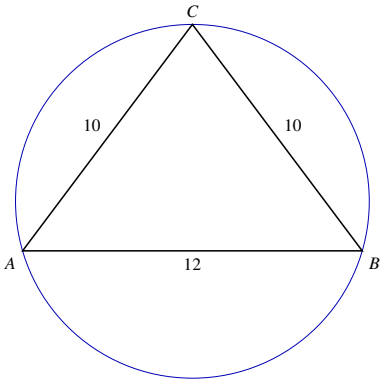
its hypotenuse is $OB = R$, a radius of the circle,

one leg is FB, half of the base,

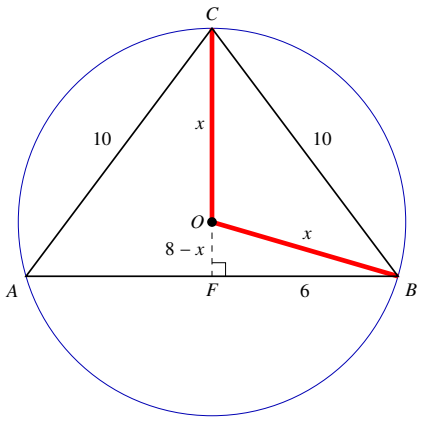
the other leg is $OF = h - R$, because $OC = R$ is also a radius.

We first compute the height h, and then we apply the Pythagorean theorem to triangle OFB.

Example 2. The sides of an isosceles triangle are 10, 10, and 12 units long. Find the radius of the circle circumscribed around the triangle.



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Solution: First we find the height. Let F be the midpoint of AB. The height CF cuts the triangle into two right triangles, and $FB = 6$. By the Pythagorean theorem in triangle FBC,

$$h^2 + 6^2 = 10^2$$
$$h^2 + 36 = 100$$
$$h^2 = 64$$

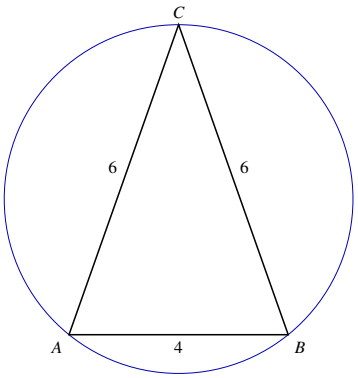
This equation has two solutions, $h = 8$ and $h = -8$. Since h is a length, it can't be negative, so $h = 8$.

Let x denote the radius. The center O is on the height CF, so $OC = OB = x$ and $OF = 8 - x$. We apply the Pythagorean theorem in the right triangle OFB.

$$6^2 + (8 - x)^2 = x^2$$
$$36 + 64 - 16x + x^2 = x^2 \qquad \text{subtract } x^2$$
$$100 - 16x = 0 \qquad \text{add } 16x$$
$$100 = 16x \qquad \text{divide by } 16$$
$$x = \frac{100}{16} = \frac{25}{4} = 6.25$$

The radius of the circumscribed circle is $\frac{25}{4} = 6.25$ units.

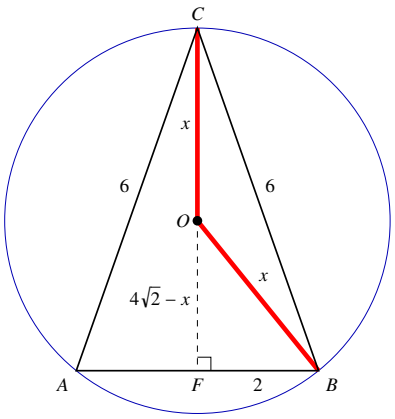
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Example 3. The sides of an isosceles triangle are 6, 6, and 4 units long. Find the exact value of the radius of the circle circumscribed around the triangle.

Solution: Again, we start with the height. Let F be the midpoint of AB, then $FB = 2$. By the Pythagorean theorem in triangle FBC,

$$h^2 + 2^2 = 6^2$$
$$h^2 + 4 = 36$$
$$h^2 = 32$$



The height is positive, so $h = \sqrt{32} = \sqrt{16 \cdot 2} = 4\sqrt{2}$. This time the height is irrational, but the method is the same.

Let x denote the radius. Then $OC = OB = x$ and $OF = 4\sqrt{2} - x$. We apply the Pythagorean theorem in the right triangle OFB. When we square $4\sqrt{2} - x$, the middle term is $2 \cdot 4\sqrt{2} \cdot x = 8\sqrt{2}\,x$, and $\left(4\sqrt{2}\right)^2 = 16 \cdot 2 = 32$.

$$2^2 + \left(4\sqrt{2} - x\right)^2 = x^2$$
$$4 + 32 - 8\sqrt{2}\,x + x^2 = x^2 \qquad \text{subtract } x^2$$
$$36 - 8\sqrt{2}\,x = 0 \qquad \text{add } 8\sqrt{2}\,x$$
$$36 = 8\sqrt{2}\,x \qquad \text{divide by } 8\sqrt{2}$$
$$x = \frac{36}{8\sqrt{2}} = \frac{9}{2\sqrt{2}}$$

We rationalize the denominator.

$$x = \frac{9}{2\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{9\sqrt{2}}{4} \approx 3.18$$

The radius of the circumscribed circle is $\dfrac{9\sqrt{2}}{4}$ units.

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Practice Problems

1. The sides of an isosceles triangle are 5, 5, and 6 units long. Find the radius of the circle circumscribed around the triangle.
2. The sides of an isosceles triangle are 17, 17, and 16 units long. Find the radius of the circle circumscribed around the triangle.
3. The sides of an isosceles triangle are 7, 7, and 4 units long. Find the exact value of the radius of the circle circumscribed around the triangle.
4. The sides of an isosceles triangle are 9, 9, and 6 units long. Find the exact value of the radius of the circle circumscribed around the triangle.
5. Find the exact value of the radius of the circle circumscribed around an equilateral triangle with sides 6 units long.
6. The legs of a right isosceles triangle are 4 units long. Find the exact value of the radius of the circle circumscribed around the triangle. Where is the center of the circle? How does this connect to the Thales circle?
7. A flower garden has the shape of an isosceles triangle, with sides 13 m, 13 m, and 10 m long. There is a lamp post at each corner of the garden. We want to build a circular walking path that passes through all three lamp posts. How long is the radius of the path? Round your answer to the nearest hundredth of a meter.
8. The sides of an isosceles triangle are 13, 13, and 24 units long. Find the radius of the circle circumscribed around the triangle. Draw a picture first: is the center of the circle inside the triangle?
9. A circle has a radius of 5 units. We draw an isosceles triangle into the circle, so that all three vertices are on the circle and the base is 8 units long. How long are the legs of the triangle? (There are two different answers.)

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Answers

1. $R = \dfrac{25}{8} = 3.125$ (the height is 4)

2. $R = \dfrac{289}{30} \approx 9.63$ (the height is 15)

3. $R = \dfrac{49}{6\sqrt{5}} = \dfrac{49\sqrt{5}}{30} \approx 3.65$ (the height is $\sqrt{45} = 3\sqrt{5}$)

4. $R = \dfrac{27}{4\sqrt{2}} = \dfrac{27\sqrt{2}}{8} \approx 4.77$ (the height is $\sqrt{72} = 6\sqrt{2}$)

5. $R = \dfrac{6}{\sqrt{3}} = 2\sqrt{3} \approx 3.46$ (the height is $\sqrt{27} = 3\sqrt{3}$)

6. $R = 2\sqrt{2} \approx 2.83$. The hypotenuse is $4\sqrt{2}$, and the height belonging to it is $2\sqrt{2}$. The center is the midpoint of the hypotenuse, so the circumscribed circle is the Thales circle of the hypotenuse.

7. $R = \dfrac{169}{24} \approx 7.04$ m (the height is 12 m)

8. $R = \dfrac{169}{10} = 16.9$ (the height is 5). The radius is longer than the height, so the center is outside of the triangle, below the base. The equation $12^2 + (5 - x)^2 = x^2$ still works, because $(5 - x)^2 = (x - 5)^2$.

9. $4\sqrt{5} \approx 8.94$ or $2\sqrt{5} \approx 4.47$. The center is 3 units from the base, so the height is $5 + 3 = 8$ or $5 - 3 = 2$.

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