Example 1. a) Consider a class of 20 students. For the next class election, students may run for president, vice president, and treasurer of the class. The election must result in 3 different students for the three positions, i.e. students cannot have multiple positions. How many different outcomes are possible on this election?

b) Once elected, the three officers must have a meeting to discuss their common agenda. In how many different ways can three students be selected to attend this meeting?

Solution: a) Let us pretend that we are selecting the students for the positions. For president we can choose any one of the 20 students. We make that choice. Now, for the vice president, we only have 19 possibilities. If we make that choice, there are only 18 left for the treasurer position. So the number of outcomes is $20 \cdot 19 \cdot 18 = \boxed{6840}$.

b) One good question is: how is this question different from the one in part a)?

In combinatorics, we will often ask ourselves two questions. One of these recurring questions is: **Does order matter?** That is precisely the difference between parts a) and b). Let us number the students 1, 2, 3,...,19 , 20. Suppose that students numbered $2, 7,$ and 18 are elected.

In part a), 2- president, $7-$ vice president, and 18- treasurer counts as a different outcome from 7-president, 18 vice president, 2 treasurer. But in part b), these two different outcomes give rise to the same attendance on the post-election meeting. Therefore, 2, 7,18 and 7,18,2 must be counted as the same outcome for the meeting. In part a), the order of the three students selected matters, while it does not matter in part b).

Let us further consider the example where students 2, 7, and 18 are elected. The number $20 \cdot 19 \cdot 18$ is not correct because it counts $2,7, 18$ several times. In fact, the number $20 \cdot 19 \cdot 18$ separately counts the cases

$2, 7, 18$	$7, 2, 18$	$18, 2, 7$
$2, 18, 7$	$7, 18, 2$	$18, 7, 2$

The overcount with students 2,7,18 is sixfold, we counted one outcome six times. As a matter of fact, we counted every possible student-triple six times. Therefore, we can adjust for this overcount by dividing by six. Indeed, $\dfrac{20 \cdot 19 \cdot 18}{6} = \boxed{1140}$ is the correct count for the possible attendance on this meeting.

Example 2. a) How many different 4-digit numbers are there using only digits 1,2,$4, 6, 7,$ and 9 with all digits different?

b) How many 4-element subsets are there of the set $A = \{1, 2, 4, 6, 7, 9\}$?

Solution: a) For the first digit, we have six choices. Because all digits must be different, there are only five choices left for the second digit. And so on, we have one less option after each choice we made. The number of outcomes is therefore $6 \cdot 5 \cdot 4 \cdot 3 = 360$.

b) This question is identical with part a), except for the role of the order. In both parts, we must choose four out of six items. In part a), the order of numbers matters, but not in part b). We can start with $6 \cdot 5 \cdot 4 \cdot 3$, but we have to adjust for the multiple counting of each outcome. Suppose our selection was 2, 6, 7, and 9. How many times did we count this outcome? This outcome was counted separately as

$2, 6, 7, 9$	$6, 2, 7, 9$	$7, 2, 6, 9$	$9, 2, 6, 7$
$2, 6, 9, 7$	$6, 2, 9, 7$	$7, 2, 9, 6$	$9, 2, 7, 6$
$2, 7, 6, 9$	$6, 7, 2, 9$	$7, 6, 2, 9$	$9, 6, 2, 7$
$2, 7, 9, 6$	$6, 7, 9, 2$	$7, 6, 9, 2$	$9, 6, 7, 2$
$2, 9, 6, 7$	$6, 9, 2, 7$	$7, 9, 2, 6$	$9, 7, 2, 6$
$2, 9, 7, 6$	$6, 9, 7, 2$	$7, 9, 6, 2$	$9, 7, 6, 2$

Because we will have a 24-fold overcount with every possible 4-element subsets, the correct count will be

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$\frac{6 \cdot 5 \cdot 4 \cdot 3}{24} = 15$. Therefore, there are $\boxed{15}$ four-element subsets of $A = \{1, 2, 4, 6, 7, 9\}$.
The overcount, 24, is the number of possible rearrangements or permutations on four objects.

Example 3. How many different four-element subsets does a set A have if it contains 10 elements?

Solution: Let us pretend first that we are creating a possible four-element subset. If the order in which we select the elements would matter, the answer would be $10 \cdot 9 \cdot 8 \cdot 7 = 5040$. But in this case, the order of selected elements does not matter, we have an overcount. Every number-quadruplet was counted as many times, as many times it could be rearranged. That number is $4! = 24$. Therefore, the correct count is $\frac{10 \cdot 9 \cdot 8 \cdot 7}{4 \cdot 3 \cdot 2 \cdot 1} = \boxed{210}$

We used $4 \cdot 3 \cdot 2 \cdot 1$ in the numerator instead of $4!$. Both numerator and denominator consist of multiplication of four numbers - representing four choices.

Definition: Let n and k be non-negative integers. The symbol $\binom{n}{k}$ (read: "n choose k") denotes the number of $k-$element subsets of an $n-$element set. Such a number is called a **combination**.

In our previous example, we have computed $\binom{10}{4}$. In addition to the notation $\binom{n}{k}$, sometimes ${}_nC_k$ is used. For example, combinations can be found under ${}_nC_k$ in our calculator.

Theorem: If n and k are non-negative integers, then $\binom{n}{k}$ can be computed as

$$\binom{n}{k} = \frac{n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot (n-k+1)}{k \cdot (k-1) \cdot (k-2) \cdot \ldots 2 \cdot 1} = \frac{n\,(n-1)\,(n-2) \cdot \ldots \cdot (n-k+1)}{k!}$$

Let us note that this formula is not always necessary. For example, $\binom{10}{1}$ is the number of one-element subsets of a ten-element set. We do not need our new formula to know that there are exactly ten ways to select one object out of 10. Or that $\binom{10}{0}$, the number of zero-element subsets is 1 because the empty set is the only zero-element subset of any set. However, the formula will always work, because both $0!$ and $1!$ are defined to be 1. Also notice that we will always get zero if k is greater than n. There is no $12-$element subset of a 10-element set. In this case, the multiplication $n\,(n-1)\,(n-2)\ldots(n-k+1)$ in the numerator of our new formula will reach the factor zero, giving us the correct answer, zero.

We perhaps also don't need the formula for combinations such as $\binom{10}{9}$ or $\binom{10}{10}$. Recall that we can count the 9-element subsets in a 10-element set by taking the entire set and select one element to remove. That can be done exactly 10 different ways, and so $\binom{10}{9} = \binom{10}{1} = 10$. There is just one $10-$ element subset of a 10-element set: the set itself.

Combinations have interesting properties. The symmetry we observed above is generally true: $\binom{n}{k} = \binom{n}{n-k}$ is true for all non-negative integers n and k. We often deploy this technique in combinatorics, and refer to it as 'counting the complement'. We will formally prove this property later. For now, we will only verify it for an example. Consider $\binom{10}{7}$.

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We will show that it is equal to $\binom{10}{3}$.

$$\binom{10}{7} = \frac{10\cdot 9\cdot 8\cdot 7\cdot 6\cdot 5\cdot 4}{7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1} = \frac{10\cdot 9\cdot 8\cdot \cancel{7\cdot 6\cdot 5\cdot 4}}{\cancel{7\cdot 6\cdot 5\cdot 4}\cdot 3\cdot 2\cdot 1} = \binom{10}{3}$$

Example 4. In how many different ways can the letters AABBBB be rearranged?

Solution: We cannot use permutations because not all objects are different. If we number the letters, A_1A_2 and A_2A_1 will just appear as AA and must be counted as a single outcome, not two. Permutation would count it as two. So, if we have no idea, we should start with systematic listing. We could start listing as

AABBBB	BAABBB	BBABAB	
ABABBB	BABABB	BBABBA	
ABBABB	BABBAB	BBBAAB	
ABBBAB	BABBBA	BBBABA	
ABBBBA	BBAABB	BBBBAA	15 cases

While we have the answer, listing is not suitable for larger numbers. Let us investigate how we found the answer. To reduce writing and also, to make the list easier to read, we could just start writing A only and leave spots empty for B, with the understanding that empty spots mean B.

A	A				
A		A			
A			A		
A				A	

⋮ and so on.

Even better, we could just note the spots selected for A. (1,2) would denote AABBBB, (3,6) would denote BBABBA

Then the complete list is

(1,2),
(1,3), (2,3),
(1,4), (2,4), (3,4),
(1,5), (2,5), (3,5), (4,5),
(1,6), (2,6), (3,6), (4,6), (5,6)

Unlike in algebra, it is typical in combinatorics to start with a notation and change to a different one while solving the problem. Discovering a better notation is often part of the problem solving, and could be the key to finding the solution. If we look at our last notation, (2,5) denotes A at spots at 2 and 5 and B at all other places, thus BABBAB.

So, what are we counting here? This list is counting the two spots we select for A. That is, selecting 2 slots out of 6. Therefore, the correct answer is $\binom{6}{2} = \frac{6\cdot 5}{2} = \boxed{15}$.

There is another form of the formula to compute combinations. Using a concrete example,

$$\binom{13}{5} = \frac{13\cdot 12\cdot 11\cdot 10\cdot 9}{5!} = \frac{13\cdot 12\cdot 11\cdot 10\cdot 9}{5!}\cdot\frac{8!}{8!} = \frac{13\cdot 12\cdot 11\cdot 10\cdot 9\cdot 8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}{5!\cdot 8!} = \frac{13!}{5!\cdot 8!}$$

In general,

$$\binom{n}{k} = \frac{n\,(n-1)\cdots(n-k+1)}{k!} = \frac{n\,(n-1)\cdots(n-k+1)}{k!}\cdot\frac{(n-k)!}{(n-k)!} = \frac{n!}{k!\cdot(n-k)!}$$

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Why is this form of the formula useful? First, it is easy to enter into a calculator, because most calculators have a factorial key. When the numbers are large, we do not have to type a long product. For example, the number of different eight-card hands from a deck of 52 cards is

$$\binom{52}{8} = \frac{52!}{8! \cdot 44!} = 752\,538\,150$$

Second, this form shows the symmetry $\binom{n}{k} = \binom{n}{n-k}$ immediately. If we replace k by $n-k$, then $n-k$ is replaced by $n-(n-k)=k$, so the two factorials in the denominator simply switch places:

$$\binom{n}{n-k} = \frac{n!}{(n-k)! \cdot k!} = \frac{n!}{k! \cdot (n-k)!} = \binom{n}{k}$$

Example 5. A pizza place offers 10 different toppings.

a) How many different pizzas can we order with exactly 3 different toppings?

b) How many different pizzas can we order with at most 3 toppings? (A plain pizza with no toppings counts too.)

Solution: a) The order of the toppings does not matter, so we count the 3-element subsets of the 10 toppings.

$$\binom{10}{3} = \frac{10 \cdot 9 \cdot 8}{3 \cdot 2 \cdot 1} = 120$$

b) A pizza with at most 3 toppings has 0, 1, 2, or 3 toppings. We add the numbers of these cases.

$$\binom{10}{0} + \binom{10}{1} + \binom{10}{2} + \binom{10}{3} = 1 + 10 + 45 + 120 = 176$$

Example 6. A club has 6 women and 5 men.

a) In how many ways can we select a committee of 2 women and 3 men?

b) In how many ways can we select a committee of 4 people that contains at least one man?

Solution: a) We make two decisions: we select 2 of the 6 women, *and* 3 of the 5 men. By the fundamental counting principle, we multiply.

$$\binom{6}{2} \cdot \binom{5}{3} = 15 \cdot 10 = 150$$

b) "At least one man" has many cases (1, 2, 3, or 4 men), but its complement, "no men at all," has only one. So we count all committees of 4 out of the 11 people, and subtract the committees made of women only.

$$\binom{11}{4} - \binom{6}{4} = 330 - 15 = 315$$

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Example 7. From a group of 12 people, we select a committee of 5, and then one of the 5 committee members is selected to be the chair. In how many ways can this be done?

Solution: First we select the committee: $\binom{12}{5} = 792$ ways. Then we select the chair from the 5 committee members: 5 ways.

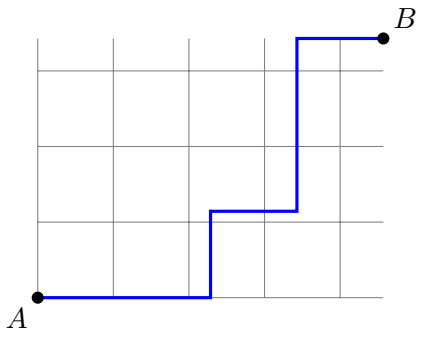
$$\binom{12}{5} \cdot 5 = 792 \cdot 5 = 3960$$

Another way: We first select the chair out of all 12 people. Then we select the other 4 committee members out of the remaining 11 people.

$$12 \cdot \binom{11}{4} = 12 \cdot 330 = 3960$$

Of course, we get the same answer.

Example 8. The picture shows a grid of streets. We walk from corner A to corner B, and we only walk to the right or up. How many different such paths are there?



Solution: Every path consists of 4 steps to the right (R) and 3 steps up (U), 7 steps in all. For example, the path shown in the picture is RRURUUR. A path is determined by which 3 of the 7 steps go up. So we select 3 out of 7 positions, just like we selected the positions of the letter A in AABBBB.

$$\binom{7}{3} = \frac{7 \cdot 6 \cdot 5}{3 \cdot 2 \cdot 1} = 35$$

Example 9. (A problem from our colleague Peter Rowell.) In how many ways can we split 6 people into two groups of three?

Solution: It is tempting to say $\binom{6}{3} = 20$: we choose 3 people for the first group, and the other 3 form the second group. But this counts every split twice. If the people are A, B, C, D, E, and F, then choosing $\{A, B, C\}$ and choosing $\{D, E, F\}$ both give the same split. So the number of splits is

$$\frac{\binom{6}{3}}{2} = \frac{20}{2} = 10$$

We can check this in another way. Let us look at the group of person A. We need to select 2 more people for this group out of the other 5, and every choice gives a different split: $\binom{5}{2} = 10$.

If the two groups have names, for example a red team and a blue team, then $\{A, B, C\}$ on the red team and $\{A, B, C\}$ on the blue team are different splits. In that case, there are $\binom{6}{3} = 20$ splits, and we do not divide by 2.

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Practice Problems

1. Compute each of the following.

   a) $\binom{20}{0}$ c) $\binom{7}{8}$ e) $\binom{6}{0}+\binom{6}{1}+\binom{6}{2}+\binom{6}{3}+\binom{6}{4}+\binom{6}{5}+\binom{6}{6}$

   b) $\binom{13}{4}$ d) $\binom{15}{14}$ f) $\binom{6}{0}-\binom{6}{1}+\binom{6}{2}-\binom{6}{3}+\binom{6}{4}-\binom{6}{5}+\binom{6}{6}$

2. Given 10 points on a circle, we connect every point with all other points. How many different triangles have all three of their vertices on the circle?

3. In the Illinois lottery, 6 numbers are selected out of 45. How many different outcomes are possible?

4. Can the handshake problem be explained in terms of combinations?

5. a) Compute $\binom{4}{0}+\binom{4}{1}+\binom{4}{2}+\binom{4}{3}+\binom{4}{4}$

   b) Consider the results from 1e) and 5a) Is there a pattern? Is it generally true? Can you find a combinatorial argument explaining this?

6. Compute each of the following. a) $\binom{3}{2}$ b) $\binom{5}{2}$ c) $\binom{7}{3}$ d) $\binom{11}{2}$

7. A pizza place offers 12 different toppings. How many different pizzas can we order with exactly 4 toppings?

8. A committee of 3 people is selected from 8 teachers and 10 students. In how many ways can this be done if the committee must have

   a) exactly one teacher, b) at least one teacher?

9. a) In how many ways can we select 3 books out of 9 to take on vacation?

   b) In how many ways can we select 6 books out of 9 to leave at home?

   c) Why are the answers to parts a) and b) the same?

10. On a grid of streets, we walk from one corner to the opposite corner, taking 5 steps to the right and 4 steps up, in some order. How many different paths are there?

11. A diagonal of a polygon connects two vertices that are not next to each other. How many diagonals does a polygon with 10 sides have?

12. An exam has 10 questions, and students must answer 7 of them.

    a) In how many ways can a student select the 7 questions?

    b) How many ways are there if the first 3 questions must be answered?

13. a) Label the following statement as true or false. *Suppose that p is a prime number, and k is an integer, $1 < k < p$. Then $\binom{p}{k}$ is a multiple of p.*

    b) Explain your answer for part a).

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14. Find n and k if we know the following. $\binom{n}{k-1} = 15\,504$, $\binom{n}{k} = 38\,760$, and $\binom{n}{k+1} = 77\,520$.

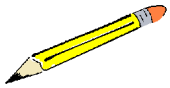
15. *Prove that for all n, k natural numbers, $\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}$



Answers

1. a) 1 b) 715 c) 0 d) 15 e) 64 f) 0 2. $\binom{10}{3} = 120$ 3. $\binom{45}{6} = 8145\,060$
4. The number of handshakes between n people is the same as the number of two-element subsets of an $n-$element set.
5. a) 16 b) Let A be a set with n elements. If we add the number of 0-element subsets, 1-element subsets, 2-element subsets, and so on, n-element subsets, then the sum is the number of subsets of the set, 2^n.
6. a) 3 b) 10 c) 35 d) 55
7. $\binom{12}{4} = 495$ 8. a) $8 \cdot \binom{10}{2} = 8 \cdot 45 = 360$ b) $\binom{18}{3} - \binom{10}{3} = 816 - 120 = 696$
9. a) $\binom{9}{3} = 84$ b) $\binom{9}{6} = 84$ c) Selecting the 3 books to take is the same as selecting the 6 books to leave at home.
10. $\binom{9}{4} = 126$ 11. $\binom{10}{2} - 10 = 45 - 10 = 35$ 12. a) $\binom{10}{7} = 120$ b) $\binom{7}{4} = 35$
13. a) It's true. b) see solutions 14. $n = 20$ and $k = 6$. 15. see solutions

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Solutions

14. Find n and k if we know the following. $\binom{n}{k-1} = 15\,504$, $\binom{n}{k} = 38\,760$ and $\binom{n}{k+1} = 77\,520$.

Solution: It is a natural instinct to want to divide these expressions by each other, because there will be significant cancellation.

$$\frac{\binom{n}{k+1}}{\binom{n}{k}} = \frac{77\,520}{38\,760} = 2 \quad \text{and} \quad \frac{\binom{n}{k}}{\binom{n}{k-1}} = \frac{38\,760}{15\,504} = \frac{5}{2}$$

$$2 = \frac{\binom{n}{k+1}}{\binom{n}{k}} = \frac{\frac{n(n-1)(n-2)\cdots(n-k)}{(k+1)!}}{\frac{n(n-1)(n-2)\cdots(n-k+1)}{k!}} = \frac{n(n-1)(n-2)\cdots(n-k)}{(k+1)!} \cdot \frac{k!}{n(n-1)(n-2)\cdots(n-k+1)}$$

$$= \frac{n(n-1)(n-2)\cdots(n-k)}{n(n-1)(n-2)\cdots(n-k+1)} \cdot \frac{k!}{(k+1)!} = \frac{n(n-1)(n-2)\cdots(n-k+1)(n-k)}{n(n-1)(n-2)\cdots(n-k+1)} \cdot \frac{k!}{(k+1)k!} = \frac{n-k}{k+1}$$

$$\frac{5}{2} = \frac{\binom{n}{k}}{\binom{n}{k-1}} = \frac{\frac{n(n-1)(n-2)\cdots(n-k+1)}{k!}}{\frac{n(n-1)(n-2)\cdots(n-k+2)}{(k-1)!}} = \frac{n(n-1)(n-2)\cdots(n-k+2)(n-k+1)}{k!} \cdot \frac{(k-1)!}{n(n-1)(n-2)\cdots(n-k+2)}$$

$$= \frac{n(n-1)(n-2)\cdots(n-k+2)(n-k+1)}{n(n-1)(n-2)\cdots(n-k+2)} \cdot \frac{(k-1)!}{k(k-1)!} = \frac{n-k+1}{k}$$

So we now have a system of equations: $\begin{cases} \frac{n-k}{k+1} = 2 \\ \frac{n-k+1}{k} = \frac{5}{2} \end{cases} \implies \begin{cases} n-k = 2(k+1) \\ 2(n-k+1) = 5k \end{cases} \implies \begin{cases} n-3k = 2 \\ 2n-7k = -2 \end{cases}$

We solve the system and get $n = 20$ and $k = 6$.

15. Claim: For all n, k natural numbers, $\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}$

proof: $\binom{n}{k} + \binom{n}{k+1} = \frac{n(n-1)(n-2)\cdots(n-k+1)}{k!} + \frac{n(n-1)(n-2)\cdots(n-(k+1)+1)}{(k+1)!}$

$$= \frac{n(n-1)(n-2)\cdots(n-k+1)}{k!} + \frac{n(n-1)(n-2)\cdots(n-k+1)(n-k)}{(k+1)!}$$

$$= \frac{n(n-1)(n-2)\cdots(n-k+1)}{k!} + \frac{n(n-1)(n-2)\cdots(n-k+1)(n-k)}{(k+1)\cdot k!}$$

$$= \frac{n(n-1)(n-2)\cdots(n-k+1)}{k!} + \frac{n(n-1)(n-2)\cdots(n-k+1)}{k!} \cdot \frac{n-k}{k+1} \quad \text{factor GCF}$$

$$= \frac{n(n-1)(n-2)\cdots(n-k+1)}{k!} \cdot \left(1 + \frac{n-k}{k+1}\right) = \frac{n(n-1)(n-2)\cdots(n-k+1)}{k!} \cdot \left(\frac{k+1+n-k}{k+1}\right)$$

$$= \frac{n(n-1)(n-2)\cdots(n-k+1)}{k!} \cdot \left(\frac{n+1}{k+1}\right) = \frac{(n+1)n(n-1)(n-2)\cdots(n-k+1)}{(k+1)k!}$$

$$= \frac{(n+1)n(n-1)(n-2)\cdots((n+1)-(k+1)+1)}{(k+1)!} = \binom{n+1}{k+1}$$

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