The **domain** of a function is the set of all numbers we are allowed to substitute into it. When a function is given by a formula and nothing else is said, its domain is the set of all real numbers for which the formula is meaningful. To find it, we look for the operations that are *not* defined for every real number, and rule out the numbers that would cause trouble.

What can go wrong. In this course, there are only four sources of trouble.

1. We can not divide by zero. **The denominator** of a fraction can not be zero.
2. **The number under a square root** (or any even root) can not be negative.
3. We can only take the logarithm of a positive number. **The argument of a logarithm** must be positive.
4. **The base of a logarithm** must be positive and different from 1.

If none of these appear in the formula, then the domain is all real numbers.

We will write the domain in two ways. **Inequality notation** describes the numbers we may substitute, such as $x \le 0$ or $x \ge 6$. **Interval notation** lists the same numbers as intervals, such as $(-\infty, 0] \cup [6, \infty)$. All real numbers is written as $(-\infty, \infty)$.

Type 1: A linear expression

Example 1. Find the domain of each of the following functions.

a) $f(x) = \dfrac{1}{x+5}$ b) $g(x) = \sqrt{x+5}$ c) $h(x) = \log_2(x+5)$

Solution: The three functions look alike, but the three operations have three different conditions.

a) The denominator can not be zero, so we need $x + 5 \neq 0$, which means $x \neq -5$. We can substitute every real number except -5.

Domain: $x \neq -5$ or $(-\infty, -5) \cup (-5, \infty)$

b) The number under the square root can not be negative, so we need $x + 5 \ge 0$, which means $x \ge -5$. Zero is allowed under a square root: $\sqrt{0} = 0$.

Domain: $x \ge -5$ or $[-5, \infty)$

c) The argument of the logarithm must be positive, so we need $x + 5 > 0$, which means $x > -5$. This time zero is not allowed: there is no exponent c with $2^c = 0$.

Domain: $x > -5$ or $(-5, \infty)$

Example 2. Find the domain of each of the following functions. a) $f(x) = \sqrt{6 - 2x}$ b) $g(x) = \ln(6 - 2x)$

Solution: a) The number under the square root can not be negative.

$$\begin{aligned} 6 - 2x &\ge 0 && \text{subtract } 6 \\ -2x &\ge -6 && \text{divide by } -2 \text{ and flip the inequality sign} \\ x &\le 3 \end{aligned}$$

Domain: $x \le 3$ or $(-\infty, 3]$

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b) The argument of the logarithm must be positive: $6 - 2x > 0$. We solve it the same way and get $x < 3$.

Domain: $x < 3$ or $(-\infty, 3)$

Type 2: A quadratic expression

For a quadratic expression we solve a quadratic inequality. We use the method of the Quadratic Inequalities handout: factor the expression, graph the parabola, and read off where the graph is above (or below) the x-axis.

Example 3. Find the domain of each of the following functions.

a) $f(x) = \dfrac{1}{x^2 - 6x}$ b) $g(x) = \sqrt{x^2 - 6x}$ c) $h(x) = \ln\left(x^2 - 6x\right)$

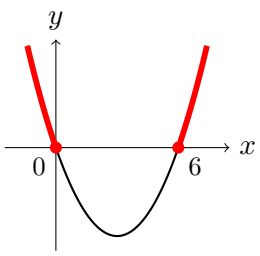
Solution: In all three parts we factor first: $x^2 - 6x = x(x - 6)$.

a) The denominator can not be zero: $x(x - 6) \neq 0$. A product is zero exactly when one of its factors is zero, so the product is zero when $x = 0$ or $x = 6$. We can substitute every other number.

Domain: $x \neq 0$ and $x \neq 6$ or $(-\infty, 0) \cup (0, 6) \cup (6, \infty)$

b) The number under the square root can not be negative: $x(x - 6) \geq 0$. We graph $y = x(x - 6)$. It is a quadratic expression with a positive leading coefficient, so its graph is an upward opening parabola, with x-intercepts at 0 and 6.

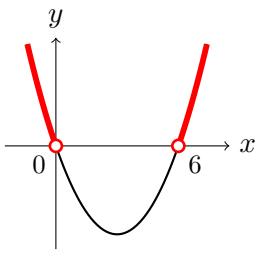
For every point of the parabola, $y = x(x - 6)$, so the inequality becomes $y \geq 0$ on the parabola. We need the points of the parabola that lie above or on the x-axis. These are the points to the left of 0 and to the right of 6, together with the x-intercepts themselves (shown in red).



$y = x(x - 6)$ for b)

Domain: $x \leq 0$ or $x \geq 6$ or $(-\infty, 0] \cup [6, \infty)$

c) The argument of the logarithm must be positive: $x(x - 6) > 0$. This is the same parabola, but now we need the points that lie strictly above the x-axis. The x-intercepts are not allowed (shown with open circles), because there $y = 0$.



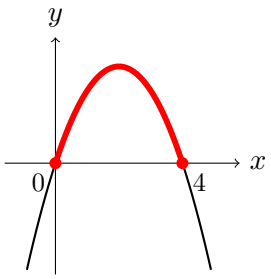
$y = x(x - 6)$ for c)

Domain: $x < 0$ or $x > 6$ or $(-\infty, 0) \cup (6, \infty)$

Example 4. Find the domain of the function $f(x) = \sqrt{4x - x^2}$.

Solution: The number under the square root can not be negative: $4x - x^2 \geq 0$. We factor: $4x - x^2 = -x(x-4)$. We do not divide by the negative leading coefficient; instead, we let it tell us the shape of the graph. The graph of $y = -x(x-4)$ is a downward opening parabola with x-intercepts at 0 and 4.

We need the points of the parabola that lie above or on the x-axis. These are the points between 0 and 4 (shown in red).



$y = -x(x-4)$

Domain: $0 \leq x \leq 4$ or $[0, 4]$

Example 5. Find the domain of each of the following functions. a) $f(x) = \dfrac{x-7}{x^2+4}$ b) $g(x) = \sqrt{x^2+4}$ c) $h(x) = \log_3\left(x^2+4\right)$

Solution: The square of a real number is never negative, so $x^2 + 4 \geq 4$ for every real number x. So $x^2 + 4$ is always positive: it is never zero, never negative, and always a good argument for a logarithm. (The graph of $y = x^2 + 4$ is a parabola that lies entirely above the x-axis.) The numerator $x - 7$ causes no trouble at all.

Domain of all three functions: all real numbers or $(-\infty, \infty)$

Type 3: The unknown is in the base of a logarithm

Example 6. Find the domain of the function $f(x) = \log_x 16$.

Solution: The argument, 16, is positive, so it causes no problem. The base must be positive and different from 1, so we need $x > 0$ and $x \neq 1$.

Domain: $x > 0$ and $x \neq 1$ or $(0,1) \cup (1, \infty)$

Why can't the base be 1? Every power of 1 is 1, so there is no exponent c with $1^c = 16$.

Example 7. Find the domain of the function $f(x) = \log_{2x-3} 7$.

Solution: The base, $2x - 3$, must be positive and different from 1. We solve both conditions.

$$\begin{array}{rcl} 2x-3 & > & 0 \\ 2x & > & 3 \\ x & > & \dfrac{3}{2} \end{array} \qquad \text{and} \qquad \begin{array}{rcl} 2x-3 & \neq & 1 \\ 2x & \neq & 4 \\ x & \neq & 2 \end{array}$$

Domain: $x > \dfrac{3}{2}$ and $x \neq 2$ or $\left(\dfrac{3}{2}, 2\right) \cup (2, \infty)$

Type 4: More than one condition

When a formula contains several of these operations, we write down *every* condition, solve each of them, and keep only the numbers that satisfy *all* of them at the same time.

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Example 8. Find the domain of the function $f(x) = \dfrac{4x-2}{6-\sqrt{x-1}}$.

Solution: The numerator causes no problem. There are two conditions.

The number under the square root can not be negative: $x - 1 \geq 0$, so $x \geq 1$.

The denominator can not be zero: $6 - \sqrt{x-1} \neq 0$.

$$\begin{array}{rcll} 6-\sqrt{x-1} & \neq & 0 & \text{add } \sqrt{x-1} \\ 6 & \neq & \sqrt{x-1} & \text{square both sides} \\ 36 & \neq & x-1 & \text{add } 1 \\ 37 & \neq & x & \end{array}$$

We need both: $x \geq 1$, but 37 is ruled out.

$$\text{Domain:} \quad x \geq 1 \text{ and } x \neq 37 \quad \text{or} \quad [1, 37) \cup (37, \infty)$$

Example 9. Find the domain of the function $f(x) = \sqrt{x+3} + \sqrt{5-x}$.

Solution: Both numbers under the square roots must be non-negative: $x + 3 \geq 0$ gives $x \geq -3$, and $5 - x \geq 0$ gives $x \leq 5$. We need both at the same time.

$$\text{Domain:} \quad -3 \leq x \leq 5 \quad \text{or} \quad [-3, 5]$$

Example 10. Find the domain of the function $f(x) = \dfrac{1}{\sqrt{9-x^2}}$.

Solution: The square root is in the denominator, so two conditions apply to it. The number under it can not be negative, *and* the square root can not be zero. Together: the number under the square root must be *positive*.

$$9 - x^2 > 0$$

We factor with the difference of squares theorem: $9 - x^2 = (3-x)(3+x) = -(x-3)(x+3)$. The graph of $y = -(x-3)(x+3)$ is a downward opening parabola with x-intercepts at -3 and 3. We need the points strictly above the x-axis: the points between -3 and 3.

$$\text{Domain:} \quad -3 < x < 3 \quad \text{or} \quad (-3, 3)$$

Example 11. Find the domain of the function $f(x) = \dfrac{3}{\log(2x - x^2)}$.

Solution: There are two conditions.

The argument of the logarithm must be positive: $2x - x^2 > 0$. We factor: $2x - x^2 = -x(x-2)$. The graph of $y = -x(x-2)$ is a downward opening parabola with x-intercepts at 0 and 2, and it is above the x-axis between them: $0 < x < 2$.

The denominator can not be zero: $\log(2x - x^2) \neq 0$. A logarithm is zero exactly when its argument is 1 (because

$10^0 = 1$), so we need $2x - x^2 \neq 1$.

$$\begin{aligned} 2x - x^2 &\neq 1 && \text{reduce one side to zero} \\ 0 &\neq x^2 - 2x + 1 && \text{factor} \\ 0 &\neq (x-1)^2 \\ x &\neq 1 \end{aligned}$$

The number 1 is between 0 and 2, so we have to rule it out.

Domain: $0 < x < 2$ and $x \neq 1$ or $(0,1) \cup (1,2)$

Summary. To find the domain of a function given by a formula,

1. look for denominators, square roots, and logarithms.
2. Write down every condition:
   - denominator $\neq 0$,
   - number under a square root ≥ 0 (and > 0 if the square root is in a denominator),
   - argument of a logarithm > 0,
   - base of a logarithm > 0 and $\neq 1$.
3. Solve each condition. For a quadratic, factor and use the parabola.
4. Keep the numbers that satisfy all conditions at the same time.
5. If there is no condition at all, the domain is all real numbers.

Practice Problems

Find the domain of each of the following functions. Give your answer in inequality notation and in interval notation.

1. $f(x) = \dfrac{1}{x-4}$
2. $f(x) = \sqrt{x-4}$
3. $f(x) = \log_2(x-4)$
4. $f(x) = \dfrac{5}{3x+6}$
5. $f(x) = \sqrt{10-2x}$
6. $f(x) = \ln(3-4x)$
7. $f(x) = \sqrt{2x+3}$
8. $f(x) = \dfrac{x+1}{5x-2}$
9. $f(x) = \dfrac{1}{x^2-9}$
10. $f(x) = \sqrt{x^2-9}$
11. $f(x) = \log_5(x^2-9)$
12. $f(x) = \dfrac{1}{4x-x^2}$
13. $f(x) = \sqrt{4x-x^2}$
14. $f(x) = \ln(x^2-5x+6)$
15. $f(x) = \sqrt{-x^2+2x+8}$
16. $f(x) = \log_3(2x^2-8x)$
17. $f(x) = \sqrt{x^2+1}$
18. $f(x) = \dfrac{x-3}{x^2+9}$
19. $f(x) = \sqrt{x^2-6x+9}$
20. $f(x) = \ln(x^2-6x+9)$
21. $f(x) = \log_x 5$
22. $f(x) = \log_{x+2} 10$
23. $f(x) = \log_{3x-1} 4$
24. $f(x) = \log_{5-x} 2$

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25. $f(x) = \sqrt{x-7} + \sqrt{7-x}$

26. $f(x) = \sqrt{x-5} - \dfrac{x+1}{x-10}$

27. $f(x) = \dfrac{1}{\sqrt{x+2}-3}$

28. $f(x) = \dfrac{\sqrt{1-x}}{\sqrt{x-5}}$

29. $f(x) = \dfrac{\sqrt{9-x^2}}{x}$

30. $f(x) = \dfrac{1}{\log_2(x-2)}$

31. $f(x) = \log_2\left(x^2-6x+8\right)$

32. $f(x) = \dfrac{1}{\log_2(x^2-6x+8)}$

33. $f(x) = \ln(x-1) + \ln(5-x)$

34. $f(x) = \dfrac{\sqrt{x-3}}{\ln(x-3)}$

Enrichment

1. Find the domain of the function $f(x) = \sqrt{x^2-1} + \sqrt{9-x^2}$.
2. Find the domain of the function $f(x) = \sqrt{\sqrt{x}-2}$.
3. Find the domain of the function $f(x) = \dfrac{\ln\left(x^2-6x\right)}{\ln\left(4-x^2\right)}$.
4. Find the domain of each of the following functions. a) $f(x) = \ln\left(x^2-6x\right) - \ln\left(4-x^2\right)$ b) $g(x) = \ln\left(\dfrac{x^2-6x}{4-x^2}\right)$
   (After Logarithms 2, these two formulas will look the same. Are the two functions the same?)
5. Write a formula for a function whose domain is $x \neq 2$ and $x \neq 5$.
6. Write a formula for a function whose domain is $[-1, 4]$.

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Answers

Practice Problems

1. $x \neq 4$, $(-\infty, 4) \cup (4, \infty)$
2. $x \geq 4$, $[4, \infty)$
3. $x > 4$, $(4, \infty)$
4. $x \neq -2$, $(-\infty, -2) \cup (-2, \infty)$
5. $x \leq 5$, $(-\infty, 5]$
6. $x < \frac{3}{4}$, $\left(-\infty, \frac{3}{4}\right)$
7. $x \geq -\frac{3}{2}$, $\left[-\frac{3}{2}, \infty\right)$
8. $x \neq \frac{2}{5}$, $\left(-\infty, \frac{2}{5}\right) \cup \left(\frac{2}{5}, \infty\right)$
9. $x \neq -3$ and $x \neq 3$, $(-\infty, -3) \cup (-3, 3) \cup (3, \infty)$
10. $x \leq -3$ or $x \geq 3$, $(-\infty, -3] \cup [3, \infty)$
11. $x < -3$ or $x > 3$, $(-\infty, -3) \cup (3, \infty)$
12. $x \neq 0$ and $x \neq 4$, $(-\infty, 0) \cup (0, 4) \cup (4, \infty)$
13. $0 \leq x \leq 4$, $[0, 4]$
14. $x < 2$ or $x > 3$, $(-\infty, 2) \cup (3, \infty)$
15. $-2 \leq x \leq 4$, $[-2, 4]$
16. $x < 0$ or $x > 4$, $(-\infty, 0) \cup (4, \infty)$
17. all real numbers, $(-\infty, \infty)$
18. all real numbers, $(-\infty, \infty)$
19. all real numbers, $(-\infty, \infty)$ (because $x^2 - 6x + 9 = (x-3)^2 \geq 0$)
20. $x \neq 3$, $(-\infty, 3) \cup (3, \infty)$
21. $x > 0$ and $x \neq 1$, $(0, 1) \cup (1, \infty)$
22. $x > -2$ and $x \neq -1$, $(-2, -1) \cup (-1, \infty)$
23. $x > \frac{1}{3}$ and $x \neq \frac{2}{3}$, $\left(\frac{1}{3}, \frac{2}{3}\right) \cup \left(\frac{2}{3}, \infty\right)$
24. $x < 5$ and $x \neq 4$, $(-\infty, 4) \cup (4, 5)$
25. only $x = 7$
26. $x \geq 5$ and $x \neq 10$, $[5, 10) \cup (10, \infty)$
27. $x \geq -2$ and $x \neq 7$, $[-2, 7) \cup (7, \infty)$
28. no real number: we would need $x \leq 1$ and $x > 5$ at the same time. The domain is empty.
29. $-3 \leq x \leq 3$ and $x \neq 0$, $[-3, 0) \cup (0, 3]$
30. $x > 2$ and $x \neq 3$, $(2, 3) \cup (3, \infty)$
31. $x < 2$ or $x > 4$, $(-\infty, 2) \cup (4, \infty)$
32. $x < 2$ or $x > 4$, and $x \neq 3 \pm \sqrt{2}$ (the argument can not be 1)
    $\left(-\infty, 3 - \sqrt{2}\right) \cup \left(3 - \sqrt{2}, 2\right) \cup \left(4, 3 + \sqrt{2}\right) \cup \left(3 + \sqrt{2}, \infty\right)$
33. $1 < x < 5$, $(1, 5)$
34. $x > 3$ and $x \neq 4$, $(3, 4) \cup (4, \infty)$

Enrichment

1. $-3 \leq x \leq -1$ or $1 \leq x \leq 3$, $[-3, -1] \cup [1, 3]$
2. $x \geq 4$, $[4, \infty)$ (we need $x \geq 0$ and $\sqrt{x} \geq 2$)
3. $-2 < x < 0$ and $x \neq -\sqrt{3}$, $\left(-2, -\sqrt{3}\right) \cup \left(-\sqrt{3}, 0\right)$ (the denominator is zero when $4 - x^2 = 1$)
4. a) $-2 < x < 0$, $(-2, 0)$ (both arguments must be positive) b) $-2 < x < 0$ or $2 < x < 6$, $(-2, 0) \cup (2, 6)$ (only the quotient must be positive: for $2 < x < 6$, the numerator and the denominator are both negative). The two functions have different domains, so they are not the same function. They agree only where both are defined.
5. For example, $f(x) = \dfrac{1}{(x-2)(x-5)}$.
6. For example, $f(x) = \sqrt{(x+1)(4-x)}$.

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