Sample Problems

1. Simplify each of the following expressions.

   a) $\dfrac{3^{2x+1}}{9^{x-1}}$ b) $\dfrac{\left(8^{b-2}\right)\left(2^{b+1}\right)}{4^{2b-3}}$ c) $5^{2x-3}\cdot 25^{1-x}$ d) $\sqrt{\dfrac{2^{3x-1}\cdot 3^{2-x}}{6^{x+1}}}$

2. Solve each of the following equations.

   a) $5^{x-2}=4$
   b) $5e^{-3x}=42$
   c) $2^{2x-3}=5^{2-x}$
   d) $5\cdot 3^{2x-1}=2^{2-x}$
   e) $3\cdot 2^{2x-3}=2\cdot 5^{x+2}$
   f) $9^x+4\cdot 3^x=5$
   g) $e^{2x}-7e^x+10=0$
   h) $4^x-2^{x+2}=-3$
   i) $9^x-3^{x+1}=4$
   j*) $9^x-4\cdot 6^x+3\cdot 4^x=0$
   k*) $4^x+200\cdot 25^x=30\cdot 10^x$

3. We placed \$1000 in a bank account with an annual compound interest rate of 8%. How long will it take until we have \$5000 in the account?

4. If we take Q amount of a certain medication, the amount of it in our system, t hours after intake is

$$A(t)=Q\cdot 0.8^t$$

   a) Approximately what percent of the mediaction is in our system 2 hours after taking it?
   b) How long until we have only 20% left in our system?
   c) How long until we have only 1% left in our system?

5. The number of cells in a sample at time t (measured in hours) is $N(t)=50\,000\ \left(1.6^{0.5t}\right)$.
   a) How many cells are there at $t=0$?
   b) How long will it take for the sample to double from the time $t=0$?
   c) How many cells are there at $t=4$?
   d) How long will it take for the sample to double from the time $t=4$?
   e) Suppose that t_1 and t_2 are given such that $N(t_2)=2N(t_1)$. Prove that the difference t_2-t_1 is constant.

Practice Problems

1. Simplify each of the following expressions so that there is at most one exponential expression is in the answer, with an exponent of x.

   a) 2^{2x+3}
   b) $5\cdot 3^{2-x}$
   c) $\dfrac{2^{x-2}}{5\cdot 3^{2x+1}}$
   d) $\dfrac{3\cdot 5^{2x+1}}{2^{4-x}}$
   e) $\dfrac{2^{x-1}\cdot 5^{x+2}}{10^{x-2}}$
   f) $\dfrac{2^{2x+1}\cdot 3^{x-1}}{6^{x-1}}$
   g) $\dfrac{5\cdot 12^{m+1}}{3^{m-1}\cdot 2^{2m+1}}$
   h) $\left(\dfrac{1}{5}\right)^{2p-3}25^{p-1}$
   i) $\sqrt{\dfrac{3^{x-1}\cdot 6^{x+2}}{2^{x-3}}}$
   j) $\sqrt{2^{10x}}\cdot\left(\dfrac{1}{8}\right)^{x-2}\cdot 4^{-x-1}$
   k) $\sqrt{\dfrac{2^{6x}\cdot 5^{8x-2}}{10^{6x+2}}}$

2. Solve each of the following equations.

a) $2 \cdot 3^{x-5} - 7 = 23$

b) $3e^{2x} - 8 = 13$

c) $4^x - 2^x - 12 = 0$

d) $\left(\frac{1}{9}\right)^x - \frac{6}{3^x} + 8 = 0$

e) $3^x + \frac{9}{3^x} = 10$

f*) $4^x - 7 \cdot 10^x + 10 \cdot 5^x = 0$

g) $5^{x-2} = 2^{2x+3}$

h*) $6 \cdot 4^x - 13 \cdot 6^x + 6 \cdot 9^x = 0$

i) $e^{2x} + e^x = 6$

j) $5 \cdot 2^{3x-1} = 3 \cdot 5^{2-x}$

k) $4 \cdot 3^{x-2} = 6^{x+1}$

l) $9^x - 3^{x+1} = 54$

m) $\frac{10^{x+2}}{2^{x-3}} = 5^{x+1}$

3. We placed \$50 in a bank account with an annual compound interest rate of 13%. How long will it take until the account contains

a) \$100 b) \$2500 c) \$1000 000

4. If we take Q amount of a certain medication, the amount of it in our system, t hours after intake is

$$A(t) = Q \cdot \left(\frac{7}{8}\right)^t$$

a) Approximately what percent of the mediaction is in our system 5 hours after taking it?
b) How long until we have 60% left in our system?
c) How long until we have only 1% left in our system?
d) How long does it take for the drug to reduce to half? (This is called the half-life of the drug.)

5. The number of cells in a sample at time t (measured in hours) is $N(t) = 100\,000\ \left(1.4^{0.3t}\right)$.
a) How many cells are there at $t = 0$?
b) How long will it take for the sample to triple from the time $t = 0$?

Sample Problems - Answers

1. a) 27 b) 2 c) $\frac{1}{5}$ d) $\frac{\sqrt{3}}{2}\left(\frac{2}{3}\right)^x$

2. a) $x = \log_5 100 = \frac{\ln 4}{\ln 5} + 2$ b) $x = \frac{1}{3}\ln\left(\frac{5}{42}\right) = \frac{\ln 8.4}{-3}$ c) $x = \log_{20} 200 = \frac{\ln 200}{\ln 20}$

   d) $\log_{18}\left(\frac{12}{5}\right) = \frac{\ln\left(\frac{12}{5}\right)}{\ln 18}$ e) $\log_{4/5}\left(\frac{400}{3}\right) = \frac{\ln\left(\frac{400}{3}\right)}{\ln\left(\frac{4}{5}\right)}$ f) $x = 0$ g) $\ln 2$ and $\ln 5$

   h) 0 and $\log_2 3$ i) $\log_3 4$ j*) 0, $\frac{\ln 3}{\ln\left(\frac{3}{2}\right)}$ k*) $\log_{2/5} 20$ and $\log_{2/5} 10$.

3. during the 21st year $\left(x = \frac{\ln 5}{\ln 1.08} \approx 20.912\,371\,88\right)$

4. a) 64% b) $\frac{\ln 0.2}{\ln 0.8} \approx 7.212\,567$ hours c) $\frac{\ln 0.01}{\ln 0.8} \approx 20.637\,7$ hours

5. a) 50 000 b) $\frac{\ln 2}{0.5 \ln 1.6} \approx 2.95$ hours c) 128 000 d) 2.95 hours e) see solutions

Practice Problems - Answers

1. a) $8 \cdot 4^x$ b) $\frac{45}{3^x}$ c) $\frac{1}{60}\left(\frac{2}{9}\right)^x$ d) $\left(\frac{15}{16}\right) \cdot 50^x$ e) 1250 f) $4 \cdot 2^x$ g) 90

   h) 5 i) $4\sqrt{6}\,(3^x)$ j) 16 k) $\frac{5^x}{50}$

2. a) $\frac{\ln 15}{\ln 3} + 5$ b) $\frac{1}{2}\ln 7$ c) 2 d) $-\frac{\ln 2}{\ln 3}, -\frac{\ln 4}{\ln 3}$ e) 0, 2 f) $\frac{\ln 2}{\ln 2 - \ln 5}, \frac{\ln 5}{\ln 2 - \ln 5}$

   g) $\frac{\ln 200}{\ln 5 - \ln 2}$ h) ± 1 i) $\ln 2$ j) $\log_{40} 30$ k) $\log_2\left(\frac{2}{27}\right)$ l) 2 m) no solution

3. a) during the 6th year $\left(x = \frac{\ln 2}{\ln 1.13} \approx 5.671\,42\right)$

   b) during the 33rd year $\left(x = \frac{\ln 50}{\ln 1.13} \approx 32.008\,663\right)$

   c) during the 82nd year $\left(x = \frac{\ln 20\,000}{\ln 1.13} \approx 81.031\,577\right)$

4. a) 51.2 909% b) $\frac{\ln 0.6}{\ln\left(\frac{7}{8}\right)}$ hours $\approx 3.825\,51$ hours c) $\frac{\ln 0.01}{\ln\left(\frac{7}{8}\right)}$ hours $\approx 34.487\,55$ hours

   d) $\frac{\ln 0.5}{\ln\left(\frac{7}{8}\right)} \approx 5.190\,9$ hours

5. a) 100 000 b) $\frac{\ln 3}{0.3 \ln 1.4}$ hours $\approx 10.883\,635$ hours

Sample Problems - Solutions

1. Simplify each of the following expressions.

a) $\dfrac{3^{2x+1}}{9^{x-1}} = \dfrac{3^{2x}\cdot 3^1}{9^x\cdot\dfrac{1}{9}} = \dfrac{\left(3^2\right)^x\cdot 3\cdot 9}{9^x} = \dfrac{9^x\cdot 27}{9^x} = 27$

b) $\dfrac{\left(8^{b-2}\right)\left(2^{b+1}\right)}{4^{2b-3}} = \dfrac{\left(\dfrac{8^b}{8^2}\right)\left(2^b\cdot 2\right)}{\left(\dfrac{4^{2b}}{4^3}\right)} = \dfrac{\left(\dfrac{8^b}{64}\right)\left(2^b\cdot 2\right)}{\left(\dfrac{4^{2b}}{64}\right)} = \left(\dfrac{8^b}{64}\right)\left(2^b\cdot 2\right)\left(\dfrac{64}{4^{2b}}\right) = \dfrac{8^b\cdot 2^b\cdot 2}{4^{2b}}$

$$= \frac{(8\cdot 2)^b\cdot 2}{\left(4^2\right)^b} = \frac{16^b\cdot 2}{16^b} = 2$$

c) $5^{2x-3}\cdot 25^{1-x} = \dfrac{5^{2x}}{5^3}\cdot\dfrac{25^1}{25^x} = \dfrac{\left(5^2\right)^x}{125}\cdot\dfrac{25}{25^x} = \dfrac{25^x}{5}\cdot\dfrac{1}{25^x} = \dfrac{1}{5}$

d) $\sqrt{\dfrac{2^{3x-1}\cdot 3^{2-x}}{6^{x+1}}} = \sqrt{\dfrac{\dfrac{2^{3x}}{2^1}\cdot\dfrac{3^2}{3^x}}{6^x\cdot 6^1}} = \sqrt{\dfrac{\dfrac{\left(2^3\right)^x}{2}\cdot\dfrac{9}{3^x}}{6^x\cdot 6}} = \sqrt{\dfrac{\dfrac{8^x}{2}\cdot\dfrac{9}{3^x}}{6^x\cdot 6}} = \sqrt{\dfrac{8^x}{2}\cdot\dfrac{9}{3^x}\cdot\dfrac{1}{6^x\cdot 6}} = \sqrt{\dfrac{9\cdot 8^x}{12\cdot 3^x\cdot 6^x}}$

$$= \sqrt{\frac{3\cdot 8^x}{4\cdot 3^x\cdot 6^x}} = \sqrt{\frac{3}{4}\cdot\frac{8^x}{3^x\cdot 6^x}} = \sqrt{\frac{3}{4}\cdot\frac{8^x}{18^x}} = \sqrt{\frac{3}{4}\cdot\left(\frac{8}{18}\right)^x} = \sqrt{\frac{3}{4}\cdot\left(\frac{4}{9}\right)^x}$$

$$= \sqrt{\frac{3}{4}\cdot\left[\left(\frac{2}{3}\right)^2\right]^x} = \sqrt{\frac{3}{4}\cdot\left(\frac{2}{3}\right)^{2x}} = \sqrt{\frac{3}{4}}\sqrt{\left[\left(\frac{2}{3}\right)^x\right]^2} = \frac{\sqrt{3}}{\sqrt{4}}\left(\frac{2}{3}\right)^x = \frac{\sqrt{3}}{2}\left(\frac{2}{3}\right)^x$$

2. Solve each of the following equations.

a) $5^{x-2} = 4$

Solution 1. We re-write the exponential expression first: $5^{x-2} = \dfrac{5^x}{5^2} = \dfrac{5^x}{25}$

$$\begin{aligned} 5^{x-2} &= 4 \\ \frac{5^x}{25} &= 4 \qquad \text{multiply by 25} \\ 5^x &= 100 \\ x &= \log_5 100 \end{aligned}$$

So the solution is $\log_5 100$. Please note that the final answer can be represented in numerous ways. For example, $\log_5 100$ can be re-written as $\dfrac{\ln 100}{\ln 5}$. It is also possible to write this number as $2+\log_5 4$ or $2+\dfrac{\ln 4}{\ln 5}$ because

$$\log_5 100 = \log_5(25\cdot 4) = \log_5 25 + \log_5 4 = 2 + \log_5 4 = 2 + \frac{\ln 4}{\ln 5}$$

All of these forms, $\log_5 100 = \dfrac{\ln 100}{\ln 5} = 2 + \log_5 4 = 2 + \dfrac{\ln 4}{\ln 5}$ are acceptable as final answer.

Solution 2.

$$\begin{aligned} 5^{x-2} &= 4 && \text{take natural logarithm} \\ \ln 5^{x-2} &= \ln 4 && \text{exponent becomes multiplier, using } \ln a^n = n \ln a \\ (x-2)\ln 5 &= \ln 4 && \text{divide by } \ln 5 \\ x-2 &= \frac{\ln 4}{\ln 5} && \text{add 2} \\ x &= \frac{\ln 4}{\ln 5} + 2 \end{aligned}$$

So the solution is $2 + \dfrac{\ln 4}{\ln 5}$. Although the results do look different from $\log_5 100$, the number we obtained using another method, $\dfrac{\ln 4}{\ln 5} + 2$ is the same number:

$$\frac{\ln 4}{\ln 5} + 2 = \frac{\ln 4}{\ln 5} + \frac{2\ln 5}{\ln 5} = \frac{\ln 4 + 2\ln 5}{\ln 5} = \frac{\ln 4 + \ln 5^2}{\ln 5} = \frac{\ln 4 \cdot 5^2}{\ln 5} = \frac{\ln 100}{\ln 5} = \log_5 100$$

b) $5e^{-3x} = 42$

Solution 1. We start with transformations: $e^{-3x} = \dfrac{1}{e^{3x}} = \dfrac{1}{\left(e^3\right)^x}$

$$\begin{aligned} 5e^{-3x} &= 42 \\ \frac{5}{\left(e^3\right)^x} &= 42 && \text{multiply by } \left(e^3\right)^x \\ 5 &= 42 \cdot \left(e^3\right)^x && \text{divide by 42} \\ \frac{5}{42} &= \left(e^3\right)^x \\ x &= \log_{\left(e^3\right)}\left(\frac{5}{42}\right) = \frac{\ln\left(\frac{5}{42}\right)}{\ln e^3} = \frac{\ln\left(\frac{5}{42}\right)}{3} = \frac{1}{3}\ln\left(\frac{5}{42}\right) \end{aligned}$$

Solution 2.

$$\begin{aligned} 5e^{-3x} &= 42 && \text{divide by 5} \\ e^{-3x} &= \frac{42}{5} && \text{take natural logarithm} \\ \ln e^{-3x} &= \ln\left(\frac{42}{5}\right) && \text{exponent becomes multiplier} \\ -3x &= \ln\left(\frac{42}{5}\right) && \text{divide by } -3 \\ x &= \frac{\ln\left(\frac{42}{5}\right)}{-3} = -\frac{1}{3}\ln\left(\frac{42}{5}\right) \end{aligned}$$

Did we get the same number using the two methods? Yes, since

$$-\frac{1}{3}\ln\left(\frac{42}{5}\right) = \frac{1}{3}(-1)\ln\left(\frac{42}{5}\right) = \frac{1}{3}\ln\left(\frac{42}{5}\right)^{-1} = \frac{1}{3}\ln\left(\frac{5}{42}\right)$$

c) $2^{2x-3} = 5^{2-x}$

Solution 1. We start with some algebraic transformations.

$$2^{2x-3} = \frac{2^{2x}}{2^3} = \frac{\left(2^2\right)^x}{2^3} = \frac{4^x}{8} \quad \text{and} \quad 5^{2-x} = \frac{5^2}{5^x} = \frac{25}{5^x}$$

$$\begin{aligned}
2^{2x-3} &= 5^{2-x} \\
\frac{4^x}{8} &= \frac{25}{5^x} && \text{multiply by 8} \\
4^x &= 8 \cdot \frac{25}{5^x} && \text{multiply by } 5^x \\
4^x \cdot 5^x &= 8 \cdot 25 && \text{use } a^n b^n = (ab)^n \\
20^x &= 200 \\
x &= \log_{20} 200 = \frac{\ln 200}{\ln 20}
\end{aligned}$$

So the only solution is $\log_{20} 200$. This number can also be written as $1 + \log_{20} 10$ or $1 + \dfrac{\ln 10}{\ln 20}$, because

$$\log_{20} 200 = \log_{20}(20 \cdot 10) = \log_{20} 20 + \log_{20} 10 = 1 + \log_{20} 10 = 1 + \frac{\ln 10}{\ln 20}$$

The final answer can be presented as either of the following: $\log_{20} 200 = \dfrac{\ln 200}{\ln 20} = 1 + \log_{20} 10 = 1 + \dfrac{\ln 10}{\ln 20}$

Solution 2.

$$\begin{aligned}
2^{2x-3} &= 5^{2-x} && \text{take natural logarithm of both sides} \\
\ln\left(2^{2x-3}\right) &= \ln\left(5^{2-x}\right) && \text{exponent becomes multiplier} \\
(2x-3)\ln 2 &= (2-x)\ln 5 && \text{distribute} \\
2(\ln 2)x - 3\ln 2 &= 2\ln 5 - (\ln 5)x && \text{add } (\ln 5)x \\
2(\ln 2)x + (\ln 5)x - 3\ln 2 &= 2\ln 5 && \text{add } 3\ln 2 \\
2(\ln 2)x + (\ln 5)x &= 2\ln 5 + 3\ln 2 && \text{factor out } x \\
x(2\ln 2 + \ln 5) &= 2\ln 5 + 3\ln 2 && \text{divide by } 2\ln 2 + \ln 5 \\
x &= \frac{2\ln 5 + 3\ln 2}{2\ln 2 + \ln 5} \\
&= \frac{\ln 5^2 + \ln 2^3}{\ln 2^2 + \ln 5} = \frac{\ln 25 + \ln 8}{\ln 4 + \ln 5} = \frac{\ln 25 \cdot 8}{\ln 4 \cdot 5} = \frac{\ln 200}{\ln 20} = \log_{20} 200
\end{aligned}$$

So $\log_{20} 200$ is the only solution.

d) $5 \cdot 3^{2x-1} = 2^{2-x}$

Solution 1. $3^{2x-1} = \dfrac{3^{2x}}{3^1} = \dfrac{\left(3^2\right)^x}{3} = \dfrac{9^x}{3}$ and $2^{2-x} = \dfrac{2^2}{2^x} = \dfrac{4}{2^x}$

$$\begin{aligned}
5 \cdot 3^{2x-1} &= 2^{2-x} \\
5 \cdot \frac{9^x}{3} &= \frac{4}{2^x} && \text{multiply by } 2^x \\
2^x \cdot 5 \cdot \frac{9^x}{3} &= 4 && \text{multiply by 3} \\
2^x \cdot 5 \cdot 9^x &= 12 && \text{divide by 5}
\end{aligned}$$

$$\begin{aligned} 2^x \cdot 9^x &= \frac{12}{5} \qquad \text{use } a^n b^n = (ab)^n \\ 18^x &= \frac{12}{5} \\ x &= \log_{18}\left(\frac{12}{5}\right) = \frac{\ln\left(\frac{12}{5}\right)}{\ln 18} = \frac{\ln 12 - \ln 5}{\ln 18} \end{aligned}$$

All three forms of the final answer are acceptable.

Solution 2.)

$$\begin{aligned} 5 \cdot 3^{2x-1} &= 2^{2-x} && \text{take the natural logarithm of both sides} \\ \ln\left(5 \cdot 3^{2x-1}\right) &= \ln\left(2^{2-x}\right) \\ \ln 5 + \ln\left(3^{2x-1}\right) &= \ln\left(2^{2-x}\right) \\ \ln 5 + (2x-1)\ln 3 &= (2-x)\ln 2 && \text{distribute} \\ \ln 5 + 2(\ln 3)x - \ln 3 &= 2\ln 2 - (\ln 2)x && \text{add } (\ln 2)x \\ \ln 5 + 2(\ln 3)x - \ln 3 + (\ln 2)x &= 2\ln 2 && \text{subtract } \ln 5 \\ 2(\ln 3)x - \ln 3 + (\ln 2)x &= 2\ln 2 - \ln 5 && \text{add } \ln 3 \\ 2(\ln 3)x + (\ln 2)x &= 2\ln 2 - \ln 5 + \ln 3 && \text{factor out } x \\ x(2\ln 3 + \ln 2) &= 2\ln 2 - \ln 5 + \ln 3 && \text{divide by } 2\ln 3 + \ln 2 \\ x &= \frac{2\ln 2 - \ln 5 + \ln 3}{2\ln 3 + \ln 2} = \frac{\ln 2^2 - \ln 5 + \ln 3}{\ln 3^2 + \ln 2} = \frac{\ln\left(\frac{12}{5}\right)}{\ln 18} \end{aligned}$$

e) $3 \cdot 2^{2x-3} = 2 \cdot 5^{x+2}$

Solution 1: $2^{2x-3} = \dfrac{2^{2x}}{2^3} = \dfrac{\left(2^2\right)^x}{8} = \dfrac{4^x}{8}$ and $5^{x+2} = 5^x \cdot 5^2 = 5^x \cdot 25$

$$\begin{aligned} 3 \cdot 2^{2x-3} &= 2 \cdot 5^{x+2} \\ 3 \cdot \frac{4^x}{8} &= 2 \cdot 5^x \cdot 25 \\ \frac{3}{8} \cdot 4^x &= 50 \cdot 5^x && \text{multiply by } 8 \\ 3 \cdot 4^x &= 400 \cdot 5^x && \text{divide by } 3 \\ 4^x &= \frac{400}{3} \cdot 5^x && \text{divide by } 5^x \\ \frac{4^x}{5^x} &= \frac{400}{3} && \text{use the rule } \frac{a^n}{b^n} = \left(\frac{a}{b}\right)^n \\ \left(\frac{4}{5}\right)^x &= \frac{400}{3} \\ x &= \log_{4/5}\left(\frac{400}{3}\right) = \frac{\ln\left(\frac{400}{3}\right)}{\ln\left(\frac{4}{5}\right)} = \frac{\ln 400 - \ln 3}{\ln 4 - \ln 5} \end{aligned}$$

$$\begin{array}{rcll}
\text{Solution 2.)} \quad 3\cdot 2^{2x-3} &=& 2\cdot 5^{x+2} & \text{take natural logarithm of both sides}\\
\ln\left(3\cdot 2^{2x-3}\right) &=& \ln\left(2\cdot 5^{x+2}\right) & \ln ab = \ln a + \ln b\\
\ln 3 + \ln\left(2^{2x-3}\right) &=& \ln 2 + \ln\left(5^{x+2}\right) & \ln a^n = n\ln a\\
\ln 3 + (2x-3)\ln 2 &=& \ln 2 + (x+2)\ln 5 & \text{distribute}\\
\ln 3 + 2(\ln 2)x - 3\ln 2 &=& \ln 2 + (\ln 5)x + 2\ln 5 & \text{subtract } (\ln 5)x\\
\ln 3 + 2(\ln 2)x - 3\ln 2 - (\ln 5)x &=& \ln 2 + 2\ln 5 & \text{subtract } \ln 3\\
2(\ln 2)x - 3\ln 2 - (\ln 5)x &=& \ln 2 + 2\ln 5 - \ln 3 & \text{add } 3\ln 2\\
2(\ln 2)x - (\ln 5)x &=& 4\ln 2 + 2\ln 5 - \ln 3 & \text{factor out } x\\
x(2\ln 2 - \ln 5) &=& 4\ln 2 + 2\ln 5 - \ln 3 & \text{divide by } 2(\ln 2) - (\ln 5)\\
x &=& \dfrac{4\ln 2 + 2\ln 5 - \ln 3}{2\ln 2 - \ln 5} & \\
x &=& \dfrac{\ln 2^4 + \ln 5^2 - \ln 3}{\ln 2^2 - \ln 5} = \dfrac{\ln\left(2^4\cdot 5^2\right) - \ln 3}{\ln\left(2^2\right) - \ln 5} = \dfrac{\ln 400 - \ln 3}{\ln 4 - \ln 5} &
\end{array}$$

f) $9^x + 4\cdot 3^x = 5$

Solution: Since $9^x = \left(3^2\right)^x = 3^{2x} = (3^x)^2$, this equation is quadratic in e^x

$$\begin{array}{rcll}
9^x + 4\cdot 3^x &=& 5 &\\
(3^x)^2 + 4\cdot 3^x - 5 &=& 0 & \text{quadratic in } 3^x\\
(3^x - 1)(3^x + 5) &=& 0 &
\end{array}$$

$$\begin{array}{rcllc}
3^x &=& 1 & \text{or} & 3^x = -5\\
x &=& \log_3 1 = 0 & & \text{no solution here}
\end{array}$$

So 0 is the only solution.

g) $e^{2x} - 7e^x + 10 = 0$

Solution: Since $e^{2x} = (e^x)^2$, this equation is quadratic in e^x.

$$\begin{array}{rcl}
e^{2x} - 7e^x + 10 &=& 0\\
(e^x)^2 - 7e^x + 10 &=& 0\\
(e^x - 2)(e^x - 5) &=& 0
\end{array}$$

$$\begin{array}{rcllc}
e^x &=& 2 & \text{or} & e^x = 5\\
x &=& \ln 2 & & x = \ln 5
\end{array}$$

So the solutions are $\ln 2$ and $\ln 5$.

h) $4^x - 2^{x+2} = -3$

Solution: Since $4^x = \left(2^2\right)^x = (2^x)^2$ and $2^{x+2} = 2^x\cdot 2^2 = 4\cdot 2^x$, this equation is quadratic in 2^x. We will introduce a new variable and solve the quadratic equation.

$$\begin{array}{rcll}
4^x - 2^{x+2} &=& -3 &\\
(2^x)^2 - 4\cdot 2^x &=& -3 & \text{Let } a = 2^x\\
a^2 - 4a &=& -3 & \text{add } 3\\
a^2 - 4a + 3 &=& 0 & \text{factor}\\
(a-1)(a-3) &=& 0 & \Longrightarrow \quad a = 1 \text{ or } a = 3
\end{array}$$

$$\begin{aligned} \text{If } a &= 1 \\ 2^x &= 1 \\ x &= 0 \end{aligned} \qquad\qquad \begin{aligned} &\text{If } a = 3 \\ &2^x = 3 \\ &x = \log_2 3 = \frac{\ln 3}{\ln 2} \end{aligned}$$

Thus the solutions are 0 and $\log_2 3$.

i) $9^x - 3^{x+1} = 4$

Solution: Since $9^x = (3^x)^2$ and $3^{x+1} = 3 \cdot 3^x$, this equation is quadratic in 3^x. We will introduce a new variable and solve the quadratic equation.

$$\begin{aligned} 9^x - 3^{x+1} &= 4 \\ (3^x)^2 - 3 \cdot 3^x &= 4 && \text{Let } a = 3^x \\ a^2 - 3a &= 4 && \text{subtract } 4 \\ a^2 - 3a - 4 &= 0 && \text{factor} \\ (a+1)(a-4) &= 0 && \Longrightarrow \quad a = 4 \quad \text{or} \quad a = -1 \end{aligned}$$

$$\begin{aligned} \text{If } a &= 4 \\ 3^x &= 4 \\ x &= \log_3 4 = \frac{\ln 4}{\ln 3} \end{aligned} \qquad\qquad \begin{aligned} &\text{If } a = -1 \\ &3^x = -1 \\ &\text{no solution here} \end{aligned}$$

So the only solution is $\log_3 4$.

j*) $9^x - 4 \cdot 6^x + 3 \cdot 4^x = 0$

Solution:

$$\begin{aligned} 9^x - 4 \cdot 6^x + 3 \cdot 4^x &= 0 \\ \left(3^2\right)^x - 4 \cdot 2^x \cdot 3^x + 3 \cdot \left(2^2\right)^x &= 0 \\ (3^x)^2 - 4 \cdot 2^x \cdot 3^x + 3 \cdot (2^x)^2 &= 0 && \text{divide by } (2^x)^2 \\ \frac{(3^x)^2}{(2^x)^2} - \frac{4 \cdot 3^x}{2^x} + 3 &= 0 \\ \left(\left(\frac{3}{2}\right)^x\right)^2 - 4\left(\frac{3}{2}\right)^x + 3 &= 0 && \text{quadratic in } \left(\frac{3}{2}\right)^x. \quad \text{Let } a = \left(\frac{3}{2}\right)^x \\ a^2 - 4a + 3 &= 0 \\ (a-3)(a-1) &= 0 && \Longrightarrow \quad a = 3 \text{ or } a = 1 \end{aligned}$$

$$\begin{aligned} \left(\frac{3}{2}\right)^x &= 3 \\ x &= \log_{3/2} 3 = \frac{\ln 3}{\ln\left(\frac{3}{2}\right)} = \frac{\ln 3}{\ln 3 - \ln 2} \end{aligned} \qquad \text{or} \qquad \begin{aligned} &\left(\frac{3}{2}\right)^x = 1 \\ &x = 0 \end{aligned}$$

So the solutions are 0 and $\log_{3/2} 3$.

k*) $4^x + 200 \cdot 25^x = 30 \cdot 10^x$

Solution: Notice first that $4^x = (2^x)^2$ and $25^x = (5^x)^2$ and $10^x = 2^x \cdot 5^x$.

$$\begin{aligned} 4^x + 200 \cdot 25^x &= 30 \cdot 10^x \\ (2^x)^2 + 200\,(5^x)^2 - 30 \cdot 2^x \cdot 5^x &= 0 \end{aligned}$$

This equation will become quadratic in $\left(\frac{2}{5}\right)^x$ if we divide it by 25^x

$$\begin{aligned}
\left(2^x\right)^2+200\left(5^x\right)^2-30\cdot 2^x\cdot 5^x &= 0 && \text{divide by } \left(5^x\right)^2\\
\frac{\left(2^x\right)^2}{\left(5^x\right)^2}+\frac{200\left(5^x\right)^2}{\left(5^x\right)^2}-\frac{30\cdot 2^x\cdot 5^x}{\left(5^x\right)^2} &= 0\\
\left(\frac{2^x}{5^x}\right)^2+200-30\left(\frac{2^x}{5^x}\right) &= 0\\
\left[\left(\frac{2}{5}\right)^x\right]^2-30\left(\frac{2}{5}\right)^x+200 &= 0 && \text{Let } a=\left(\frac{2}{5}\right)^x\\
a^2-30a+200 &= 0\\
(a-20)(a-10) &= 0 && \Longrightarrow\quad a=20 \quad\text{or}\quad a=10
\end{aligned}$$

$$\begin{aligned}
\text{If } a &= 20\\
\left(\frac{2}{5}\right)^x &= 20\\
x &= \log_{2/5} 20=\frac{\ln 20}{\ln\left(\frac{2}{5}\right)}
\end{aligned}$$

$$\begin{aligned}
&\text{If } a=10\\
&\left(\frac{2}{5}\right)^x=10\\
&x=\log_{2/5} 10=\frac{\ln 10}{\ln\left(\frac{2}{5}\right)}
\end{aligned}$$

So the equation has two solutions, $\log_{2/5} 20$ and $\log_{2/5} 10$.

3. We placed \$1000 in a bank account with an annual compound interest rate of 8%. How long will it take until we have \$5000 in the account?

   Solution: Let x denote the time it takes to reach \$5000.

$$\begin{aligned}
1000\cdot 1.08^x &= 5000\\
1.08^x &= 5\\
\ln 1.08^x &= \ln 5
\end{aligned}$$

$$x\ln 1.08=\ln 5$$
$$x=\frac{\ln 5}{\ln 1.08}\approx 20.912\,371\,88$$

   Thus our account will reach \$5000 sometime during the 21st year.

4. If we take Q amount of a certain medication, the amount of it in our system, t hours after intake is

$$A(t)=Q\cdot 0.8^t$$

   a) Approximately what percent of the mediaction is in our system 2 hours after taking it?
   Solution: We substitute $t=2$ into the formula

$$A(2)=Q\cdot 0.8^2=0.64Q=64\%\text{ of } Q$$

   b) How long until we have only 20% left in our system?
   Solution: We have to find t for which $A(t)=0.2Q$

$$\begin{aligned}
A(t) &= 0.2Q\\
Q\cdot 0.8^t &= 0.2Q && \text{divide by } Q\\
0.8^t &= 0.2 && \text{take natural logarithm of both sides}\\
\ln 0.8^t &= \ln 0.2 && \text{exponent becomes a multiplier}\\
t\ln 0.8 &= \ln 0.2 && \text{divide by } \ln 0.8\\
t &= \frac{\ln 0.2}{\ln 0.8}\approx 7.212\,567
\end{aligned}$$

A little more than 7 hours, 7.212567 hours.

c) How long until we have only 1% left in our system?
Solution: We have to find t for which $A(t) = 0.01Q$

$$\begin{aligned}
A(t) &= 0.01Q \\
Q \cdot 0.8^t &= 0.01Q && \text{divide by } Q \\
0.8^t &= 0.01 && \text{take natural logarithm of both sides} \\
\ln 0.8^t &= \ln 0.01 && \text{exponent becomes a multiplier} \\
t \ln 0.8 &= \ln 0.01 && \text{divide by } \ln 0.8 \\
t &= \frac{\ln 0.01}{\ln 0.8} \approx 20.6377
\end{aligned}$$

5. The number of cells in a sample at time t (measured in hours) is $N(t) = 50\,000\ \left(1.6^{0.5t}\right)$.
a) How many cells are there at $t = 0$?
Solution: We substitute $t = 0$ into the formula $N(t) = 50\,000\ \left(1.6^{0.5t}\right)$.

$$N(0) = 50\,000\ \left(1.6^{0.5(0)}\right) = 50\,000\ \left(1.6^0\right) = 50\,000 \cdot 1 = 50\,000$$

b) How long will it take for the sample to double from the time $t = 0$?
Solution: We have to find t for which $N(t) = 2N(0)$

$$\begin{aligned}
N(t) &= 2N(0) && N(0) = 50\,000 \\
N(t) &= 2 \cdot 50\,000 && N(t) = 50\,000\ \left(1.6^{0.5t}\right) \\
50\,000\ \left(1.6^{0.5t}\right) &= 2 \cdot 50\,000 && \text{divide by } 50\,000 \\
1.6^{0.5t} &= 2 && \text{take natural logarithm} \\
\ln\left(1.6^{0.5t}\right) &= \ln 2 && \text{exponent becomes multiplier} \\
0.5t \ln 1.6 &= \ln 2 && \text{divide by } 0.5 \ln 1.6 \\
t &= \frac{\ln 2}{0.5 \ln 1.6} \approx 2.94954
\end{aligned}$$

It will take 2.94954 hours for the sample to double.

c) How many cells are there at $t = 4$?
Solution: We substitute $t = 4$ into the formula $N(t) = 50\,000\ \left(1.6^{0.5t}\right)$.

$$N(4) = 50\,000\ \left(1.6^{0.5(4)}\right) = 50\,000\ \left(1.6^2\right) = 50\,000 \cdot 2.56 = 128\,000$$

d) How long will it take for the sample to double from the time $t = 4$?
Solution: We have to find t for which $N(t) = 2N(4)$

$$\begin{aligned}
N(t) &= 2N(4) && N(4) = 1280\,000 \\
N(t) &= 2 \cdot 128\,000 && N(t) = 50\,000\ \left(1.6^{0.5t}\right) \\
50\,000\ \left(1.6^{0.5t}\right) &= 256\,000 && \text{divide by } 50\,000 \\
1.6^{0.5t} &= \frac{256\,000}{50\,000} && \text{simplify} \\
1.6^{0.5t} &= \frac{128}{25} && \text{take natural logarithm fo both sides}
\end{aligned}$$

$$\begin{aligned}
\ln\left(1.6^{0.5t}\right) &= \ln\left(\frac{128}{25}\right) && \text{exponent becomes multiplier} \\
0.5t\ln 1.6 &= \ln\left(\frac{128}{25}\right) && \text{divide by } 0.5\ln 1.6 \\
t &= \frac{\ln\left(\frac{128}{25}\right)}{0.5\ln 1.6} \approx 6.949\,54
\end{aligned}$$

It will be at $t \approx 6.949\,54$ hours when the sample is twice as much as at $t = 4$ hour. Thus it takes $6.949\,54 - 4 = 2.949\,54$ hours for the sample to double.

e) Suppose that t_1 and t_2 are given such that $N(t_2) = 2N(t_1)$. Prove that the difference $t_2 - t_1$ is constant.

$$\begin{aligned}
N(t_2) &= 2N(t_1) && N(t) = 50\,000\left(1.6^{0.5t}\right) \\
50\,000\left(1.6^{0.5t_2}\right) &= 2\cdot 50\,000\left(1.6^{0.5t_1}\right) && \text{divide by } 50\,000 \\
1.6^{0.5t_2} &= 2\cdot\left(1.6^{0.5t_1}\right) && \text{divide by } 1.6^{0.5t_1} \\
\frac{1.6^{0.5t_2}}{1.6^{0.5t_1}} &= 2 && \text{use } \frac{a^n}{a^m} = a^{n-m} \\
1.6^{0.5t_2-0.5t_1} &= 2 && \text{factor out } 0.5 \\
1.6^{0.5(t_2-t_1)} &= 2 && \text{take natural logarithm} \\
\ln\left(1.6^{0.5(t_2-t_1)}\right) &= \ln 2 && \text{use } \ln a^n = n\ln a \\
0.5(t_2-t_1)\ln 1.6 &= \ln 2 && \text{divide by } 0.5\ln 1.6 \\
t_2 - t_1 &= \frac{\ln 2}{0.5\ln 1.6} \approx 2.949\,54
\end{aligned}$$

This means that no matter how much time is lapsed, or how many cells are in the sample, it will always take the same amount of time for it to double. This is an interesting and important property of exponential functions.