

Recall our growing number system. We start with the natural numbers ($\mathbb{N}$), then extend to the integers ($\mathbb{Z}$), then to the rational numbers ($\mathbb{Q}$), and finally to the real numbers.

$$\mathbb{N} \rightarrow \mathbb{Z} \rightarrow \mathbb{Q} \rightarrow \mathbb{R}$$

This is what we hoped would happen to our concept and notation of exponents. We started with a definition that established exponentiation as repeated multiplication and expanded the definition to first the integers, then to the rational numbers. We still hope to go further and define any real exponent.

The step from natural numbers to integers was an algebraic one - we just added new numbers to $\mathbb{N}$ until our set was closed under subtraction. If we thought about 2^{-3} as 2^{4-7} then we could easily define 2^{-3} to be the result of the division $\frac{2^4}{2^7}$. Similarly, stepping out to the rational numbers was an algebraic one. Rational numbers are all quotients of integers. For every rational number x , there exists an integer so that when x is multiplied by it, the result is an integer. For example, $\frac{2}{7}$ becomes an integer if we multiply it by 7. Therefore, we could think of $5^{2/7}$ as a number that results in 5^2 when raised to the 7th power. Given the algebraic nature of these expansions, the definition of exponents was easy to extend.

Stepping out from the rational numbers to the real numbers was a much more messy, non-algebraic step. We can explain $\mathbb{Z}$ and $\mathbb{Q}$ in terms of natural numbers and the basic operations. Recall that $\mathbb{Z}$ is the closure of $\mathbb{N}$ under subtraction, and $\mathbb{Q}$ is the closure of $\mathbb{Z}$ under division. The set of all rational numbers is closed under addition, subtraction, multiplication, and division. So why did we even need to enlarge $\mathbb{Q}$ further? Recall that what brought π , a famous irrational number into our life was the desire that our numbers should be able to represent distances, or points on a line. As it turns out, describing distances by numbers is very different from starting with the counting numbers and operations on them. For example, π would be half of the circumference of a circle that has radius we labeled as 1. We described the real numbers as filling up the number line - that there are as many real numbers as many points are there on a line.

This is not an algebraic step and so in case of exponents, stepping out from $\mathbb{Q}$ to $\mathbb{R}$ would also be a non-algebraic step.

Before we can see how real exponents were defined, we need to understand more how we got from rational to real.

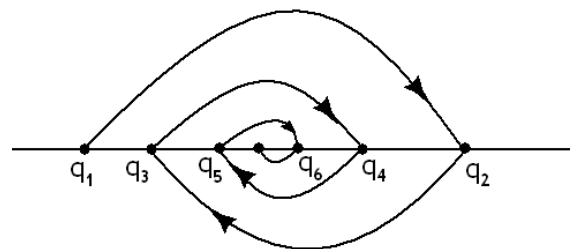
A New Operation

Consider the Fibonacci Sequence 1, 1, 2, 3, 5, 8, 13, 21, 34, ... We will create another sequence, $\{q_n\}$ that is formed by taking the quotients of consecutive term in the Fibonacci sequence. That is, $\frac{1}{1}, \frac{2}{1}, \frac{3}{2}, \frac{5}{3}, \frac{8}{5}, \frac{13}{8}, \dots$

$$q_n = \frac{F_{n+1}}{F_n} \quad \text{for all natural number } n.$$

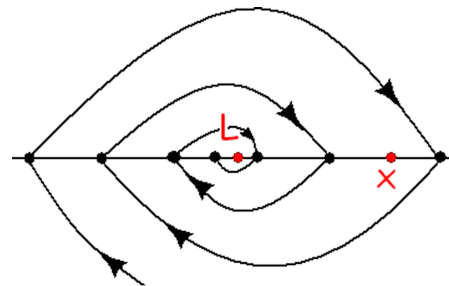
The decimal presentations of the first few terms in this sequence show an interesting pattern.

$$\begin{array}{ll} q_1 = \frac{F_2}{F_1} = \frac{1}{1} = 1 & \frac{F_6}{F_5} = \frac{8}{5} = 1.6 \\ q_2 = \frac{F_3}{F_2} = \frac{2}{1} = 2 & \frac{F_7}{F_6} = \frac{13}{8} = 1.625 \\ q_3 = \frac{F_4}{F_3} = \frac{3}{2} = 1.5 & \frac{F_8}{F_7} = \frac{21}{13} \approx 1.6153846 \\ q_4 = \frac{F_5}{F_4} = \frac{5}{3} \approx 1.66 & \frac{F_9}{F_8} = \frac{34}{21} \approx 1.61904762 \end{array}$$



The value of these quotients jumps back and forth, and every jump is smaller than the one before. This has an interesting consequence. As the value of n grows larger and larger, the values of q_n seem to be closer and closer to each other.

We can show that there is a unique number L to which this sequence gets closer and closer and closer.



This number L would be a number that is **always** inside the shrinking swirl, no matter how far we are in the sequence. And this number would be the **only** such number. If we take *any other number* x , the sequence eventually closes down on L so close so that x is outside the swirl then and after that forever. Therefore, the sequence is no longer getting closer and closer to x . Not all sequences, but some of the sequences approach a unique number L . We call this number **the limit** of the sequence.

Irrational Exponents

We will see an algebraic and also a visual approach to irrational exponents. These two arguments are essentially the same.

One way of thinking of the set of real numbers has to do with limits. We take the set of all rational numbers, create every possible sequence (that's a lot of sequences!). Some of these will have a limit, some will not. If we combine the rational numbers with all the limits of all such sequences, we get the set of all real numbers.

The algebraic approach: Suppose we would like to define something like $5^{\sqrt{2}}$. From our calculator, the decimal presentation of $\sqrt{2}$ begins as

$$\sqrt{2} \approx 1.414\,213\,562\,373\,10$$

We don't know how to handle irrational exponents, but we do know how to deal with rational exponents. Recall that, $a^{b/c} = \sqrt[c]{a^b}$ or $(\sqrt[c]{a})^b$. Consider a sequence of longer and longer, and also finer and finer decimal approximations of $\sqrt{2}$.

$a_1 = 1$ This sequence has a limit. The greater
 $a_2 = 1.4$ n is, the closer a_n is to $\sqrt{2}$.

$$5^1 = 5$$

$$5^{1.4} = 5^{\frac{14}{10}} = \left(\sqrt[10]{5}\right)^{14} \approx 9.518\,269\,693\,58$$

$a_3 = 1.41$ On the other hand, all terms of the
 $a_4 = 1.414$ sequence are rational numbers, so we
 $a_5 = 1.4142$ know how to use them as exponents.

$$5^{1.41} = 5^{\frac{141}{100}} = \left(\sqrt[100]{5}\right)^{141} = 9.672\,699\,728\,93$$

$$5^{1.414} = 5^{\frac{1414}{1000}} = \left(\sqrt[1000]{5}\right)^{1414} = 9.735\,171\,039\,2$$

$$5^{1.4142} = 5^{\frac{14142}{10000}} = \left(\sqrt[10000]{5}\right)^{14142} = 9.738\,305\,17$$

$$a_6 = 1.41421$$

$\vdots$ As the numbers a_n in the exponent approach $\sqrt{2}$, the results of the 5^{a_n} get closer and closer to a unique number.

We used one of the infinitely many different sequences of numbers with their limit being $\sqrt{2}$. No matter which one we use, we always get the same limit. We define this number to be $5^{\sqrt{2}}$.

A visual approach We started to extend the world of exponents, because we wanted to use 2^x without having to worry about the value of x . Before we define irrational exponents visually, it is useful to know two important facts about the real numbers.

Theorem: Suppose that a and b are real numbers such that $a < b$. Then there exists a rational number x with $a < x < b$. In short: every interval contains a rational number. We also say sometimes that the set of all rational numbers is **dense**.

Theorem: Suppose that a and b are real numbers such that $a < b$. Then there exists an irrational number y with $a < y < b$. In short: every interval contains an irrational number. We also say that the set of all irrational numbers is **dense**.

Both rational and irrational numbers are dense. Between any two numbers, there is always a rational and also an irrational number. Suppose x is a real number. There is no such thing as the closest rational (or irrational) number to x . There is always more such numbers closer to x .

Suppose we would like to graph $y = 5^x$ on $\mathbb{Q}$ as its domain, but using the regular coordinate system. What would the graph look like?

If a is rational, then 5^a is defined and so $(a, 5^a)$ is a point on the graph. Therefore, the graph would be defined on infinitely many points. If b is irrational, then 5^b is not (yet) defined and so the graph of $y = 5^x$ will have a gap at $x = b$. This graph therefore, will also have infinitely many gaps. Both rationals and irrationals are dense. So, no matter what interval we select, the graph of $y = 5^x$ will have a point and a gap over the interval. Because of all the gaps caused by the irrational numbers, the graph will never connect two points.

The graph wouldn't contain a connected line, only separate points. Also, we would also see many points everywhere on the graph. Between two rational numbers, there is always a third one. So, the dots are everywhere: there is no point 'closest' to another one on the graph. Also, the points on the graph would not be all over the place. They would align along a single curve. We connect all the points on the graph without lifting our pencil. Because there are so many points on the graph already defined, there is just one curve that will contain all these points. For all numbers x (rational and irrational) now there is a point of this curve with that x -coordinate. We define the y -coordinate of this point to be 5^x .

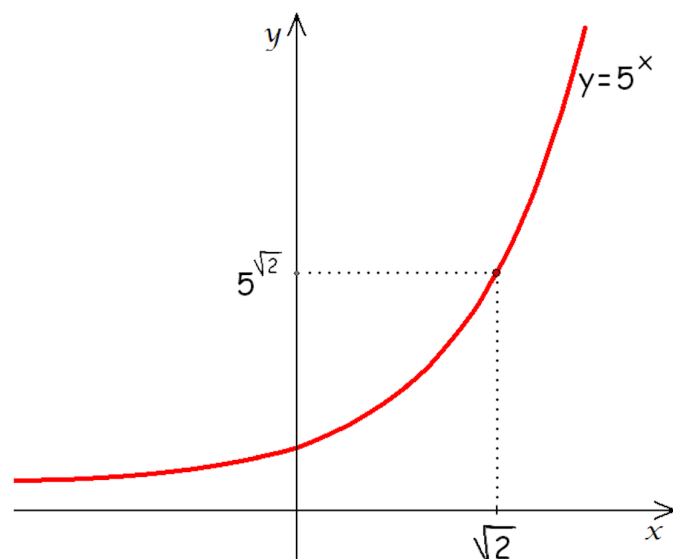
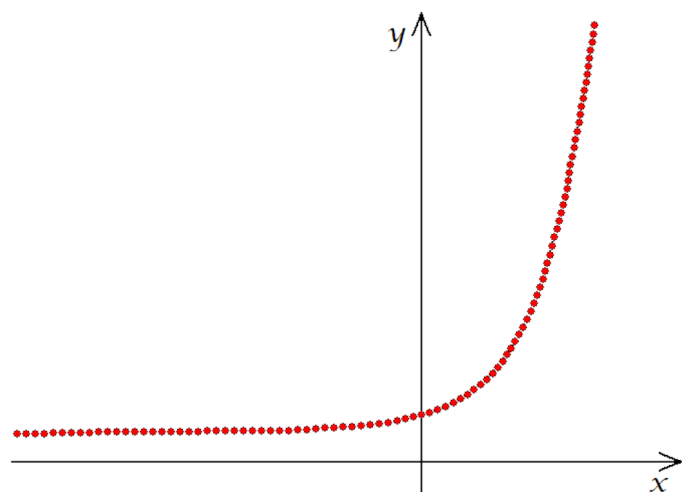
We have achieved our goal to meaningfully define 5^x for all real number x . Also note that the rules of exponents remain the same.

$y = 5^x$ is our first expression graphed with the variable in the exponent. We will often use the term **exponential** to refer to this.

Exercises

True or false? Prove or disprove each of the following.

1. If a and b are both rational, then a^b is also rational.
2. If a and b are both irrational, then a^b is also irrational.





Answers

1. Prove or disprove: If a and b are both rational, then a^b is also rational.

Absolutely not! 5 and $\frac{1}{2}$ are both rational but $5^{1/2} = \sqrt{5}$ is irrational.

2. Prove or disprove: If a and b are both irrational, then a^b is also irrational.

This is not true. It is possible to have an irrational base and exponent, yet the result is rational. Let us consider first $(\sqrt{5})^{\sqrt{2}}$. We don't know if this number is rational.

If this number is rational, we are done, we have found an example. If this number is irrational, then let us raise it to the power of $\sqrt{2}$.

$$\left((\sqrt{5})^{\sqrt{2}} \right)^{\sqrt{2}} = (\sqrt{5})^{\sqrt{2} \cdot \sqrt{2}} = (\sqrt{5})^2 = 5$$

Either way, we found an example.