Let $(a_n) = a_1, a_2, a_3, ...$ be a geometric sequence. If we denote the first element a_1 by a and the common ratio by r, then the nth element a_n can be computed as

$$a_n = ar^{n-1}$$

and the sum of the first nth elements $s_n = a_1 + a_2 + a_3 + ... + a_n$ can be computed as

$$\begin{aligned} s_n &= na && \text{if } r = 1 \\ s_n &= a\frac{1-r^n}{1-r} = a\frac{r^n-1}{r-1} && \text{if } r \neq 1 \end{aligned}$$

If $-1 < r < 1$, the the infinite sum $s = a_1 + a_2 + a_3 + ...$ also exists and can be computed as

$$s = \frac{a}{1-r}$$

Sample Problems

1. Let (a_n) be a geometric sequence with $a_1 = -3$ and $r = -2$. Compute each of the following.

   a) a_9 b) a_{10} c) s_{15} d) s_{18} e) Find n if $a_n = -196\,608$

2. A ship is 120 miles from land when its engine breaks. In the first hour, the ship covers 40 miles. After that, the engine loses 20% of its capacity in every hour.

   a) How long would it take for the ship to travel 75 miles with the broken engine?

   b) What is the greatest distance the ship could cover with the broken engine?

3. Assume an annual compound interest rate of 5%, compounded annually.

   a) Find the present value of a payment of \$1000, paid out a year from today.

   b) Find the present value of a payment of \$1000, paid out ten years from today.

   c) Find the present value of ten annual payments of \$1000, starting with the first payment right now.

   d) Find the present value of infinitely many annual payments of \$1000, starting with the first payment right now.

© Hidegkuti, Powell, 2009

Practice Problems

1. Let (a_n) be a geometric sequence with $a_1 = 64\,000$ and $r = \dfrac{1}{2}$. Compute each of the following.

   a) a_5 b) a_{14} c) s_{10} d) s_{15} e) Find n if $a_n = 125$.

2. A ship is 900 miles from land when its engine breaks. In the first hour, the ship covers 60 miles. After that, the engine loses 5% of its capacity in every hour.

   a) How long would it take for the ship to travel 750 miles with the broken engine?

   b) What is the greatest distance the ship could cover with the broken engine?

3. Assume an annual compound interest rate of 6%, compounded annually.

   a) Find the present value of a payment of \$1500, paid out a year from today.

   b) Find the present value of a payment of \$1500, paid out five years from today.

   c) The present value of a payment of \$1500, paid out n years from today is approximately \$467.71. Find n.

   d) Find the present value of twenty annual payments of \$1500, starting with the first payment right now.

   e) Find the present value of infinitely many annual payments of \$1000, starting with the first payment right now.

Sample Problems - Answers

1. a) -768 b) 1536 c) $-32\,769$ d) $262\,143$ e) 17

2. a) a little more than two hours b) 200 miles

3. a) \$952.38 b) \$613.91 c) \$8107.82 d) \$21 000

Practice Problems - Answers

1. a) 4000 b) $\dfrac{125}{16} = 7.812\,5$ c) $127\,875$ d) $127996.093\,75$ e) 10

2. a) $\dfrac{\ln \frac{3}{8}}{\ln 0.95} \simeq 19.121\,98$ it would take a little more than nineteen hours b) 1200 miles

3. a) \$1415.09 b) \$1120.89 c) 20 d) \$18 237.17 e) \$26 500

Sample Problems - Solutions

1. Let (a_n) be a geometric sequence with $a_1 = -3$ and $r = -2$. Compute each of the following.

a) $a_9 = a_1 r^8 = -3(-2)^8 = -768$

b) $a_{10} = a_1 r^9 = -3(-2)^9 = 1536$

c) Compute $s_{15} = a\dfrac{1-r^{15}}{1-r} = -3\dfrac{1-(-2)^{15}}{1-(-2)} = -32\,769$

d) Compute $s_{18} = a\dfrac{1-r^{18}}{1-r} = -3\dfrac{1-(-2)^{18}}{1-(-2)} = 262\,143$

e) Find n if $a_n = -196\,608$.
Solution: Solve $-3(-2)^x = -196\,608$ for x.

$$\begin{aligned} -3(-2)^x &= -196\,608 \\ (-2)^x &= 65\,536 \end{aligned}$$

Becasue logarithms only work with positive numbers, we need to notice that x has to be even and in this case $(-2)^x = 2^x$. After this, we can solve this equation using logarithms.

$$\begin{aligned} 2^x &= 65\,536 \\ \ln 2^x &= \ln 65\,536 \\ x\ln 2 &= \ln 65\,536 \\ x &= \frac{\ln 65\,536}{\ln 2} = 16 \end{aligned}$$

Since $x = 16$, this means $a_n = a_1 r^{16}$ and so this is the 17th element.

2. A ship is 120 miles from land when its engine breaks. In the first hour, the ship covers 40 miles. After that, the engine loses 20% of its capacity in every hour.

a) How long would it take for the ship to travel 75 miles with the broken engine?
Solution: We first find the formula for s_n, given the data above. Since the engine loses 20% in each hour, the engine's capacity decreases to its 80% in every hour. Thus, $a = 40$ and $r = 0.8$.

$$s_n = a\frac{1-r^n}{1-r} = 40\frac{1-0.8^n}{1-0.8} = 40\frac{1-0.8^n}{0.2} = \frac{40}{0.2}(1-0.8^n) = 200(1-0.8^n)$$

$$\begin{aligned} 75 &= s_n \\ 75 &= 200(1-0.8^n) && \text{solve for } n \\ \frac{75}{200} &= 1-0.8^n && 1-\frac{75}{200} = 0,625 \\ 0.8^n &= 0.625 \\ \ln 0.8^n &= \ln 0.625 \\ n\ln 0.8 &= \ln 0.625 \\ n &= \frac{\ln 0.625}{\ln 0.8} \simeq 2.106\,3 \end{aligned}$$

It would take a little bit more than two hours to travel 75 miles.

b) What is the greatest distance the ship could cover with the broken engine?
Solution: Consider the formula $s_n = 200(1-0.8^n)$ As n becomes greater and greater, 0.8^n aproaches zero, and so $1-0.8^n$ approaches 1, thus s_n approaches 200. The ship will eventually cover any distance up to 200 miles. (It will never actually go as far as 200 miles.)

3. Assume an annual compound interest rate of 5%, compounded annually.

a) Find the present value of a payment of \$1000, paid out a year from today.
Solution: We need to find out how much money to put in a bank account with a 5% interest rate, so that it becomes \$1000 in a year.

$$\begin{aligned} 1.05x &= \$1000 \\ x &= \frac{\$1000}{1.05} \simeq \$952.38 \end{aligned}$$

b) Find the present value of a payment of \$1000, paid out ten years from today.
Solution: We need to find out how much money to put in a bank account with a 5% interest rate, so that it becomes \$1000 in ten years.

$$\begin{aligned} 1.05^{10}x &= \$1000 \\ x &= \frac{\$1000}{1.05^{10}} \simeq \$613.91 \end{aligned}$$

c) Find the present value of ten annual payments of \$1000, starting with the first payment right now.
Solution: We need to realize that the present value of these future payments form a geometric sequence. The first present value is \$1000, the second is $\frac{\$1000}{1.05}$, the third is $\frac{\$1000}{1.05^2}$ and so on, this geometric sequence has its first element $a = \$1000$ and common ratio $r = \frac{1}{1.05}$. We are asked to compute s_{10}. We will first find an expression for s_n.

$$\begin{aligned} s_n &= a\frac{1-r^n}{1-r} = \$1000\frac{1-\left(\frac{1}{1.05}\right)^n}{1-\frac{1}{1.05}} = \$1000\frac{1-\left(\frac{1}{1.05}\right)^n}{\frac{1.05-1}{1.05}} = \$1000\frac{1-\left(\frac{1}{1.05}\right)^n}{\frac{0.05}{1.05}} \\ &= \$1000\left(\frac{1.05}{0.05}\right)\left(1-\left(\frac{1}{1.05}\right)^n\right) = \$1000\,(21)\left(1-\left(\frac{1}{1.05}\right)^n\right) = \$21\,000\left(1-\left(\frac{1}{1.05}\right)^n\right) \end{aligned}$$

We now substitute $n = 10$.

$$s_n = \$21\,000\left(1-\left(\frac{1}{1.05}\right)^{10}\right) \simeq \$8107.82$$

d) Find the present value of infinitely many annual payments of \$1000, starting with the first payment right now.

Solution: Consider the formula $s_n = \$21\,000\left(1-\left(\frac{1}{1.05}\right)^n\right)$ As n becomes greater and greater, $\left(\frac{1}{1.05}\right)^n$ aproaches zero, and so $1-\left(\frac{1}{1.05}\right)^n$ approaches 1, thus s_n approaches 21 000.

© Hidegkuti, Powell, 2009