Sample Problems

1. Plot the graph of $f(x) = -5(x+1)x^2(x-2)^3$.
2. Plot the graph of $f(x) = (-2-x)^2(x+1)(4-x)^2(x-2)(6-3x)(-x+1)$.

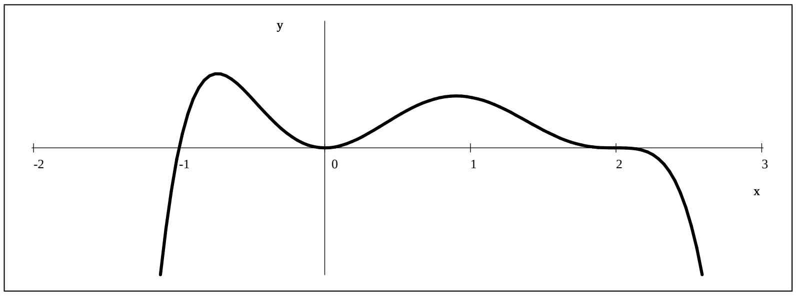
Practice Problems

Plot the graph of each of the following functions.

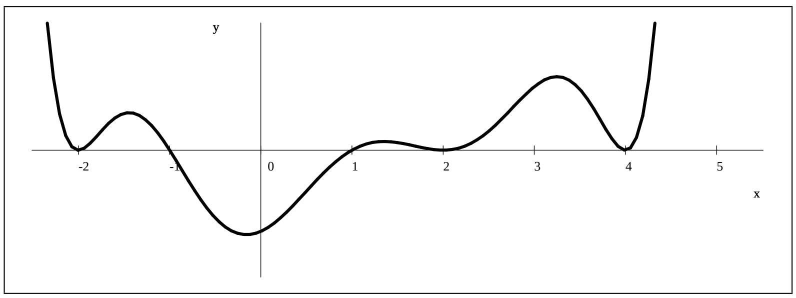
1. $f(x) = (x-3)^2(x+1)$
2. $f(x) = (x+4)(x+1)^2(x-3)$
3. $f(x) = -2(x+1)(1-x)^2(2-x)^3(6-2x)$
4. $f(x) = -(x+6)(x+5)(x+3)(x+1)^2(x-3)^2(x-6)(x-7)$
5. $f(x) = (x+1)(4-x)^2(x-2)x^3(6-3x)(-x+1)$
6. $f(x) = (x+2)(x+1)^2x(x-1)^4(x-2)^3$
7. $f(x) = (3x+24)(x+5)(x+8)(x+1)(5-x)^2(7-x)$
8. $f(x) = -(x+2)(x+1)^3x^2(x-1)(x-2)^2$
9. $f(x) = -(x+1)(3-x)^2(x-2)x^3(8-2x)(-x+1)$
10. $f(x) = -3(x-5)(2x+6)\left(4-x^2\right)^2(2x+10)(x-3)$
11. $f(x) = -2(x+2)(x+1)^2x^3(x-2)^5(x-3)^4$

Sample Problems - Answers

1. $f(x) = -5(x+1)x^2(x-2)^3$

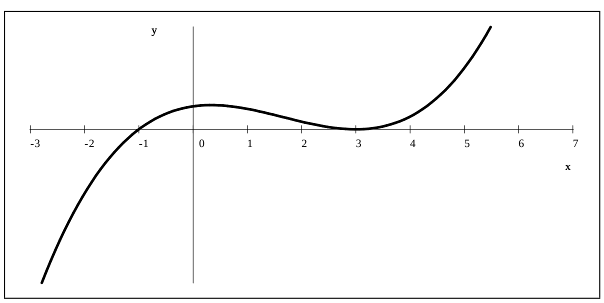


2. $f(x) = (-2-x)^2(x+1)(4-x)^2(x-2)(6-3x)(-x+1)$.

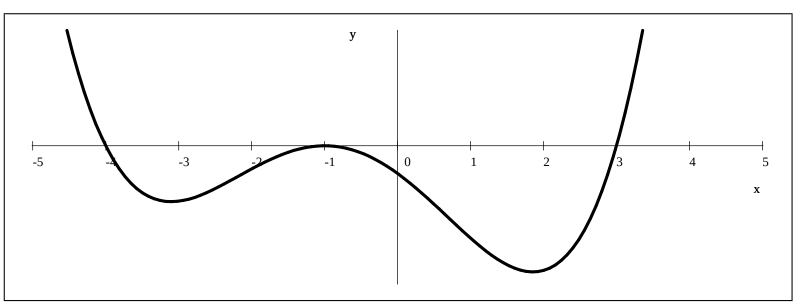


Practice Problems - Answers

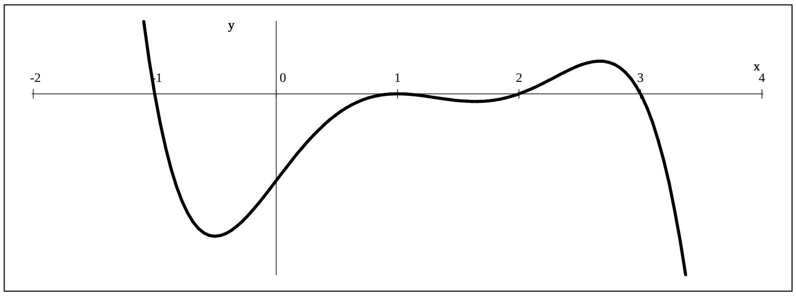
1. $f(x) = (x-3)^2(x+1)$



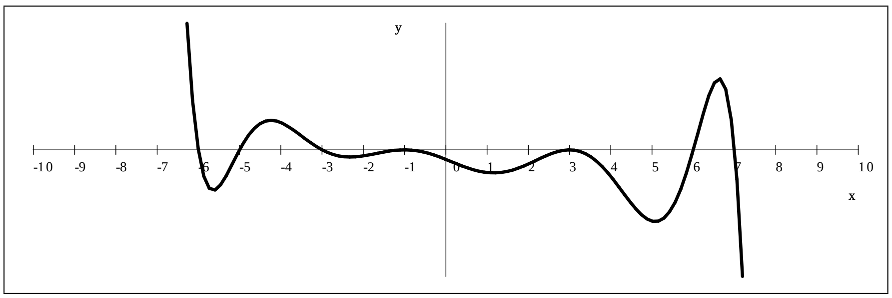
2. $f(x) = (x+4)(x+1)^2(x-3)$



3. $f(x) = -2(x+1)(1-x)^2(2-x)^3(6-2x)$



4. $f(x) = -(x+6)(x+5)(x+3)(x+1)^2(x-3)^2(x-6)(x-7)$
Solution: Since it is a polynomial, the functions domain is $\mathbb{R}$, and f is continuous on $\mathbb{R}$. The degree is 9, the leading coefficient is negative, and so the end-behavior is $\lim\limits_{x\to-\infty} f(x) = \infty$ and $\lim\limits_{x\to\infty} f(x) = -\infty$
$x-$intercepts: at $x = -6, -5, -3, -1, 3, 6$ and 7, and nowhere else. There is a change in sign at $x = -6, -5, -3, 6,$ and 7. There is no change in sign at $x = -1$ and 3.



5. $f(x) = (x+1)(4-x)^2(x-2)x^3(6-3x)(-x+1)$
Solution: Since it is a polynomial, the functions domain is $\mathbb{R}$, and f is continuous on $\mathbb{R}$.
We first bring this expression to a form easier to graph by factoring out the leading coefficient from each factor.

$$\begin{aligned}(4-x)^2 &= (x-4)^2 \quad \text{since} \\ (4-x)^2 &= [-1(x-4)]^2 = (-1)^2(x-4)^2 = (x-4)^2\end{aligned}$$

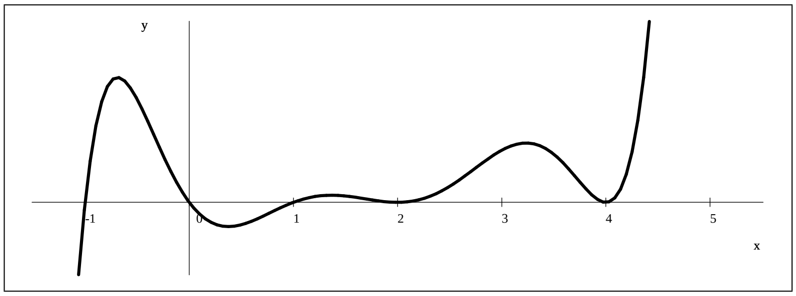
$$\begin{aligned}6-3x &= -3(x-2) \quad \text{and} \\ (-x+1) &= -(x-1)\end{aligned}$$

The function's equation is then

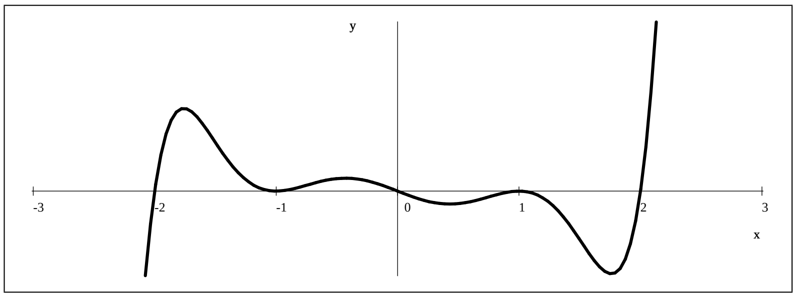
$$\begin{aligned}(x+1)(4-x)^2(x-2)x^3(6-3x)(-x+1) &= \quad \text{factor leading coefficients} \\ (x+1)(x-4)^2(x-2)x^3(-3)(x-2)(-1)(x-1) &= \quad \text{bring numbers to front} \\ 3(x+1)(x-4)^2(x-2)x^3(x-2)(x-1) &= \quad \text{organize factors left-to right by zero}\end{aligned}$$

$$= 3(x+1)x^3(x-1)(x-2)^2(x-4)^2$$

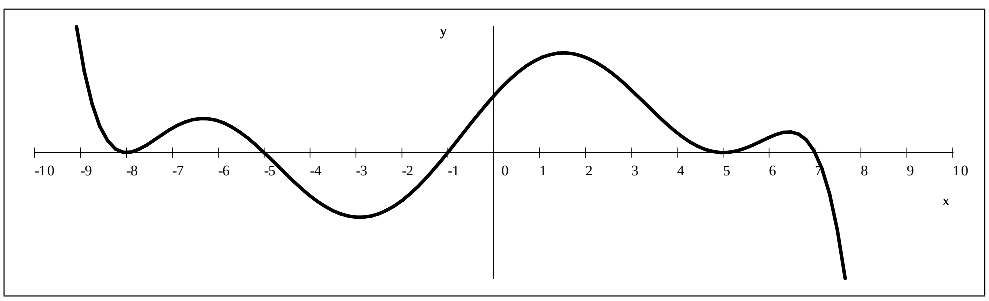
We can now determine the degree of the polynomial and the sign of its leading coefficient. (degree: 9, positive leading coefficient.) Based on this, the end-behavior is: $\lim\limits_{x\to-\infty} f(x) = -\infty$ and $\lim\limits_{x\to\infty} f(x) = \infty$
$x-$intercepts: at $x = -1, 0, 1, 2$ and 4, and nowhere else. There is a change in sign at $x = -1, 0,$ and 1. There is no change in sign at $x = 2$ and 4.



6. $f(x) = (x+2)(x+1)^2 x (x-1)^4 (x-2)^3$
   Solution: We first determine the degree: $1+2+1+4+3 = 11$. Sine the leading coefficient is positive, we have that $\lim\limits_{x\to-\infty} f(x) = -\infty$ and $\lim\limits_{x\to\infty} f(x) = \infty$. The rest of the story is given to us by continuity. The $x-$intercepts are $-2, -1, 0, 1,$ and 2. Looking at the multiplicity of the roots, we can determine if f changes sign around the zero or not.



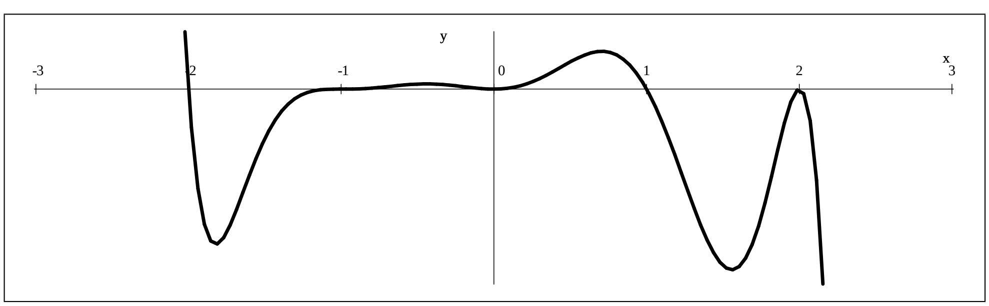
7. $f(x) = (3x+24)(x+5)(x+8)(x+1)(5-x)^2(7-x)$



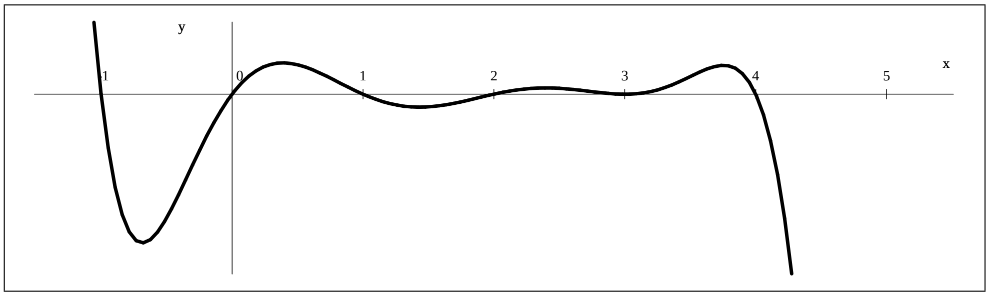
8. $f(x) = -(x+2)(x+1)^3 x^2 (x-1)(x-2)^2$.
   Solution: The degree is 9, the leading coefficient is negative Thus the end-behavior is:

$$\lim_{x\to-\infty} = \infty \quad \text{and} \quad \lim_{x\to\infty} = -\infty$$

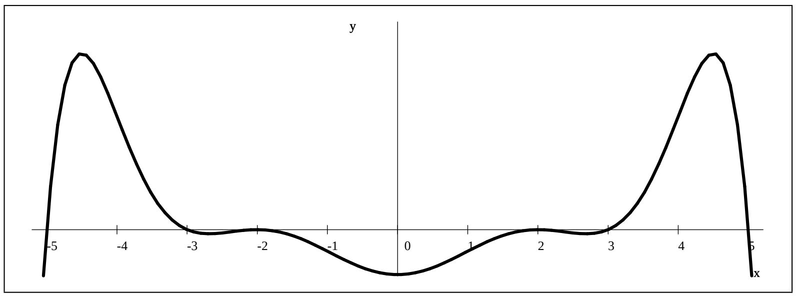
   There is sign change through the zeroes corresponding to linear factors with odd exponents: at -2, -1, and 1. There is no change of sign through the zeroes corresponding to linear factors with even exponents: at 0. The graph is continuous, thus we have:



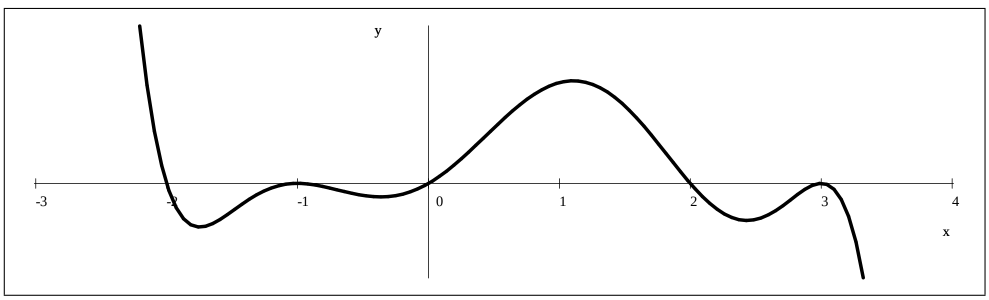
9. $f(x) = -(x+1)(3-x)^2(x-2)x^3(8-2x)(-x+1)$



10. $f(x) = -3(x-5)(2x+6)\left(4-x^2\right)^2(2x+10)(x-3)$

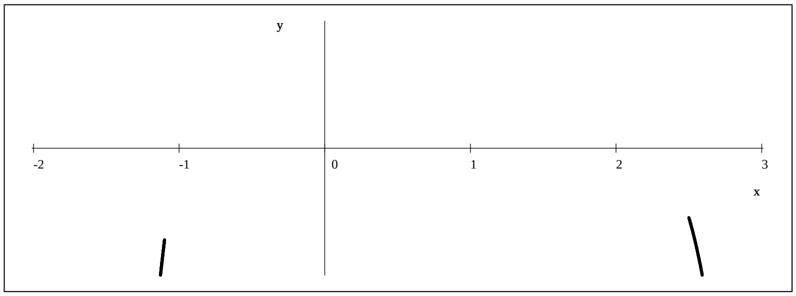


11. $f(x) = -2(x+2)(x+1)^2x^3(x-2)^5(x-3)^4$.
Solution: degree: 15 (odd), leading coefficient negative

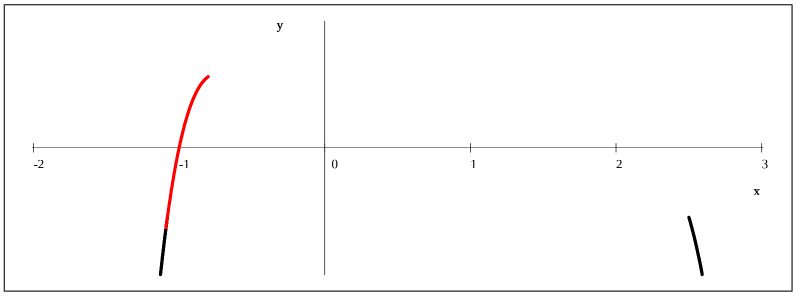


Sample Problems - Solutions

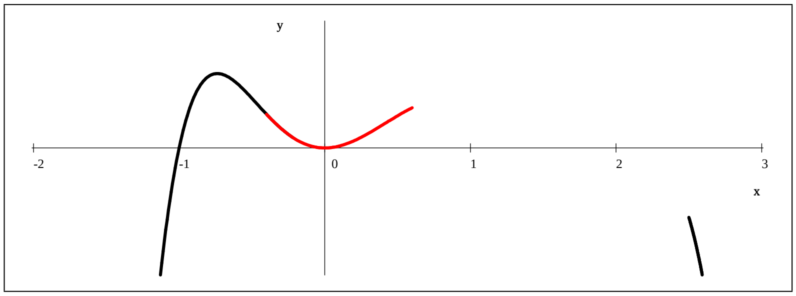
1. Plot the graph of $f(x) = -5(x+1)x^2(x-2)^3$
Solution: Since f is a polynomial, its domain is the set of all real numbers (in short: $\mathbb{R}$) and f is continuous on the entire domain.
We determine that f is of degree 6, and its leading coefficient is negative. Based on this, the end-behavior is
$$\lim_{x\to-\infty} f(x) = -\infty \quad \text{and} \quad \lim_{x\to\infty} f(x) = -\infty$$
and so our graph so far is



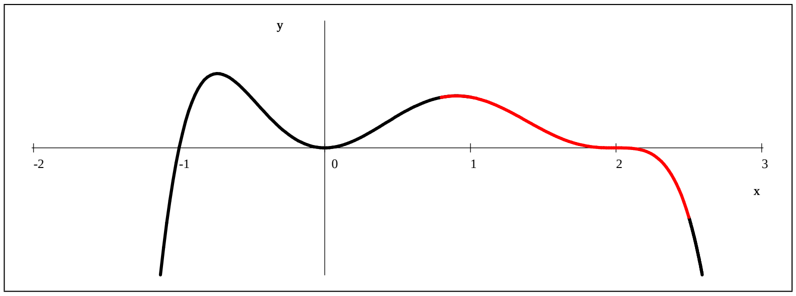
The $x-$intercepts are at $x = -1, 0,$ and 2, and there are no other zeroes. We will determine the behavior of each zero, left to right.
First, at $x = -1$. At $x = -1$, the factor $(x+1)$ is zero. Since this linear factor is raised to the first power (odd power), there is a change in sign there. Thus our graph is now



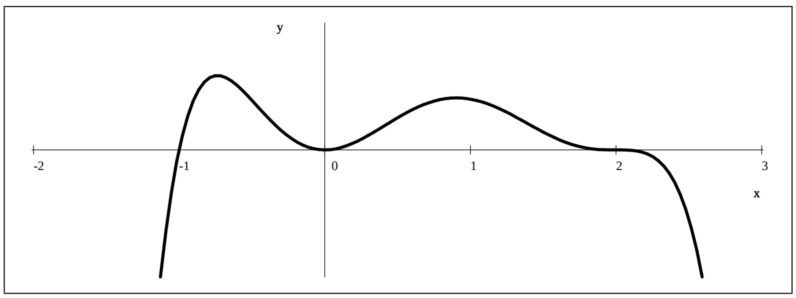
The next zero is at $x = 0$. At $x = 0$, the factor x is zero. Since this linear factor is raised to the second power (even power), there is no change in the sign there. Thus our graph is now



The next zero is at $x = 2$. At $x = 2$, the factor $x - 2$ is zero. Since this linear factor is raised to the third power (odd), there is a change in the sign there. Thus our graph is now



and so the final graph is



Note that we can not yet determine how tall the "humps" are between the zeroes. We can only conclude that they exist because the function is continuous.

2. Plot the graph of $f(x) = (-2-x)^2 (x+1)(4-x)^2 (x-2)(6-3x)(-x+1)$
Solution: First we bring the polynomial to a form that is more convenient for graphing.
First, we will factor out the leading coefficients from each of the linear factors.

$$(-2-x)^2 = (-x-2)^2 = [-1(x+2)]^2 = (-1)^2 (x+2)^2 = (x+2)^2$$

In short, $(-2-x)^2 = (x+2)^2$. Similarly, $(4-x)^2 = (x-4)^2$ and $(6-3x) = -3(x-2)$ and $(-x+1) = -(x-1)$. Thus the function f is

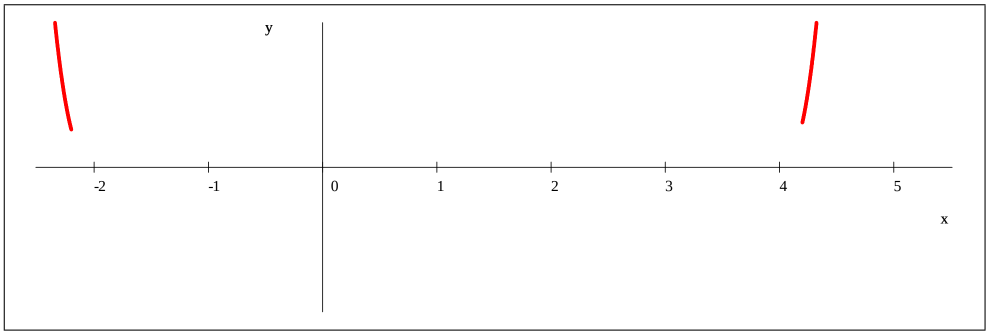
$$\begin{aligned} f(x) &= (-2-x)^2 (x+1)(4-x)^2 (x-2)(6-3x)(-x+1) \\ &= (x+2)^2 (x+1)(x-4)^2 (x-2)(-3)(x-2)(-1)(x-1) \end{aligned}$$

We now re-arrange the factors: we bring the numbers to the front, and oraganize the linear factors by their zeroes, left to right.

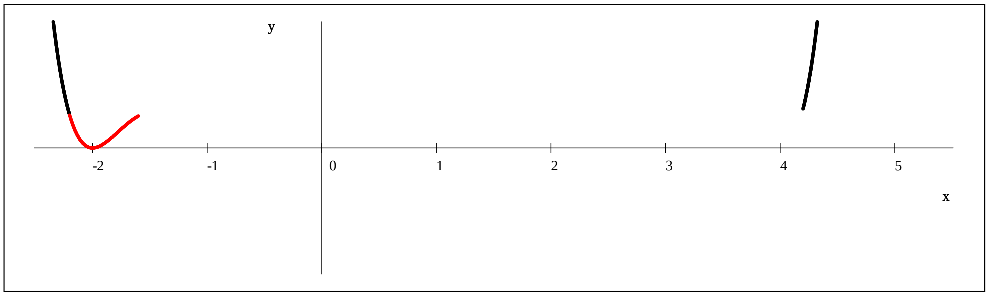
$$f(x) = 3(x+2)^2 (x+1)(x-1)(x-2)^2 (x-4)^2$$

We will use this form to graph the function. The degree and the sign of the leading coefficient determines the end-behavior of f. The degree is 8, the leading coefficient is positive, and so $\lim_{x\to-\infty} f(x) = \lim_{x\to\infty} f(x) = \infty$. The polynomial has zeroes only at $x = -2$, -1, 1, 2, and 4, and at no other points. Since all polynomials are continuous on $\mathbb{R}$, change in sign can only occur through a zero.

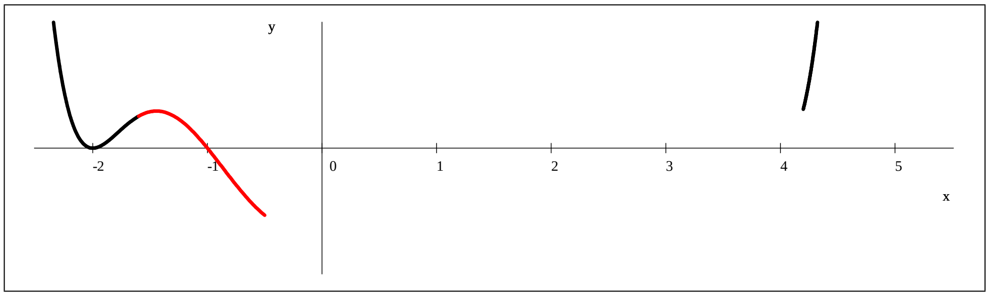
We graph left to right, starting and ending with positive values as discussed above.



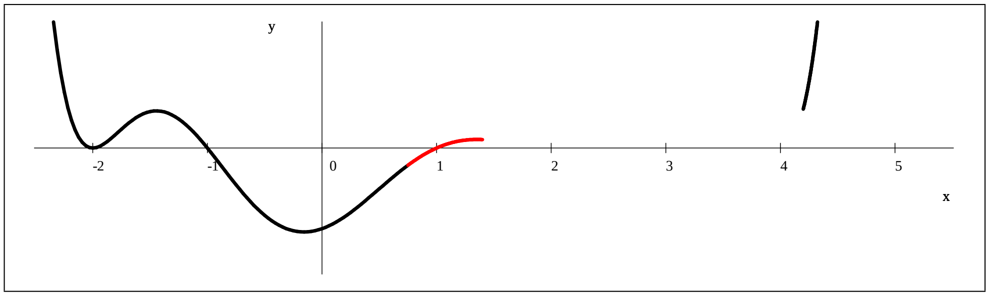
The linear factor $(x+2)$ has multiplicity 2, (its exponent is 2). Consequently, f will not change sign between numbers less than 2 and numbers larger than 2. (Very much like the function $g(x)=(x+2)^2$ at $x=-2$)



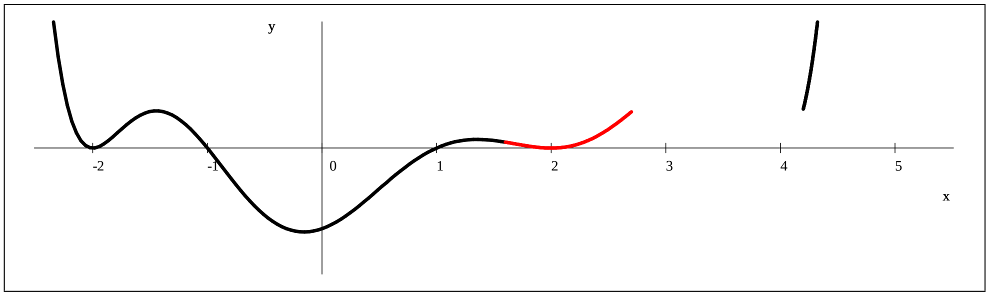
The next zero is at -1, caused by the linear factor $(x+1)$. It is not repeated (exponent 1) and thus f changes sign between numbers less than and greater than -1.



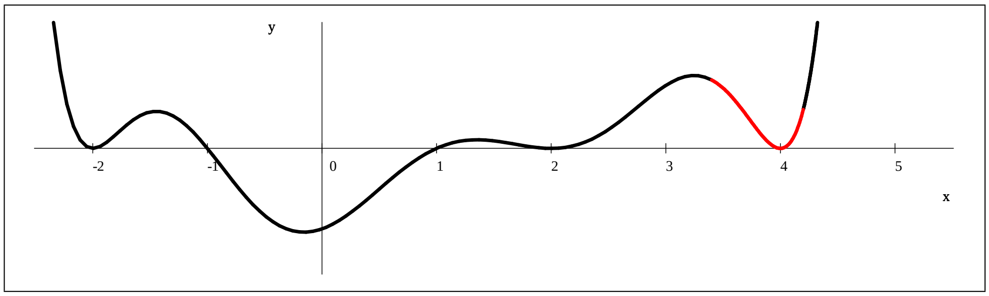
The next zero is caused by the factor $(x-1)$, thus there is again a change in sign.



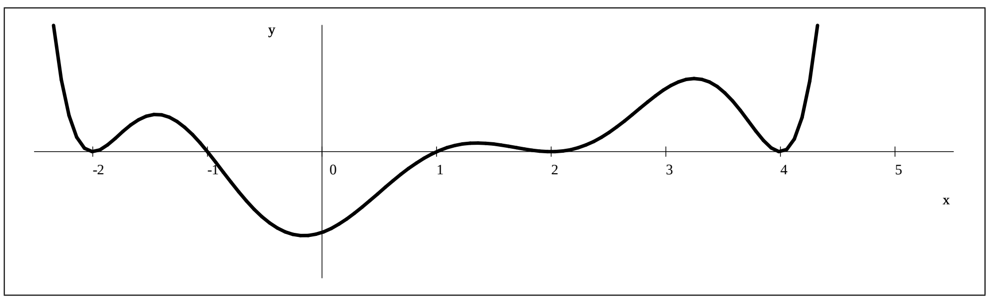
The next zero is caused by $(x-2)^2$. Because this exponent is even, there is no change in sign.



The next zero is caused by $(x-4)^2$. Because this exponent is even, there is no change in sign.



Thus the graph is:



Note that we can not yet determine how tall the "humps" are between the zeroes. We can only conclude that they exist because the function is continuous.

