# Sample Problems

Solve each of the following inequalities.

1. $10x + x^2 < -21$
2. $2x + 3 \leq x^2$
3. $x^2 - 10x + 20 \geq -9$
4. $6x - x^2 \geq 0$
5. $x^2 > 8x - 11$
6. $\frac{1}{2}x^2 - 3x + 5 \leq \frac{1}{2}$

# Practice Problems

Solve each of the following inequalities.

1. $2x + x^2 \geq 35$
2. $-12x - 2x^2 > 20$
3. $2x^2 - 4 < 7x$
4. $(x + 3)^2 \leq 25$
5. $x^2 - 14x \leq -49$
6. $x^2 - 10x \geq 1$
7. $x^2 + 30x + 1 > 4x - 170$

# Answers

## Sample Problems

1. inequality notation: $-7 < x < -3$
   interval notation: $(-7, -3)$
2. inequality notation: $x \leq -1$ or $x \geq 3$
   interval notation: $(-\infty, -1] \cup [3, \infty)$
3. $\mathbb{R}$ (all real numbers are solution)
   interval notation: $(-\infty, \infty)$
4. inequality notation: $0 \leq x \leq 6$
   interval notation: $[0, 6]$
5. inequality notation: $x < 4 - \sqrt{5}$ or $x > 4 + \sqrt{5}$
   interval notation: $\left(-\infty, 4 - \sqrt{5}\right) \cup \left(4 + \sqrt{5}, \infty\right)$
6. $x = 3$ in interval notation: $[3, 3]$

## Practice Problems

1. inequality notation: $x \leq -7$ or $x \geq 5$
   interval notation: $(-\infty, -7] \cup [5, \infty)$
2. There is no solution
3. inequality notation: $-\frac{1}{2} < x < 4$
   interval notation: $\left(-\frac{1}{2}, 4\right)$
4. inequality notation: $-8 \leq x \leq 2$
   interval notation: $[-8, 2]$
5. $x = 7$
6. $x \leq 5 - \sqrt{26}$ or $x \geq 5 + \sqrt{26}$
   interval notation: $\left(-\infty, 5 - \sqrt{26}\right] \cup \left[5 + \sqrt{26}, \infty\right)$
7. $\mathbb{R}$ (all numbers are solution)

# Sample Problems Solutions

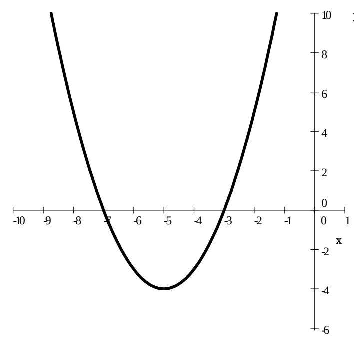
1. Solve the inequality $10x + x^2 < -21$

Solution: We reduce one side to zero first and then factor. (There are several factoring techniques possible, we will factor by completing the square.)

$$\begin{aligned} 10x + x^2 &< -21 \\ x^2 + 10x + 21 &< 0 \\ \underbrace{x^2 + 10x + 25} - 25 + 21 &< 0 \\ (x+5)^2 - 4 &< 0 \\ (x+5+2)(x+5-2) &< 0 \\ (x+7)(x+3) &< 0 \end{aligned}$$

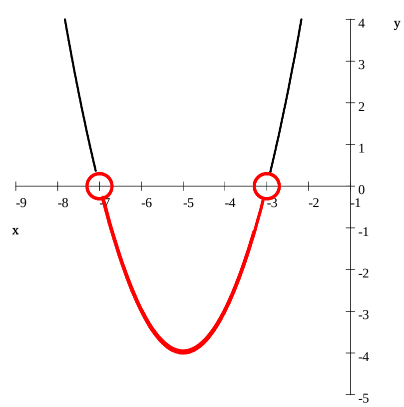
$$(x+5)^2 = x^2 + 10x + 25$$

The left-hand side is a quadratic expression with a positive leading coefficient. The graph of such an expression is a regular (or upward opening) parabola, with $x$-intercepts at $x = -7$ and $x = -3$. We plot the graph of $y = (x+7)(x+3)$.



$y = (x+7)(x+3)$

The inequality to be solved is $(x+7)(x+3) < 0$. For every point of the parabola, $y = (x+7)(x+3)$. So the inequality $(x+7)(x+3) < 0$ becomes $y < 0$ on the parabola. So we need to find the $x-$values for which $y$ is negative. That is the $x-$coordinates of all points on the parabola that lie below the $x-$axis. That's easy: it's the part between the $x-$intercepts. The $x-$coordinate of these points range from $-7$ to 3. Thus the solution is: $-7 < x < 3$ . The same result-in interval notation, $\boxed{(-7, 3)}$.



$y = (x+7)(x+3)$

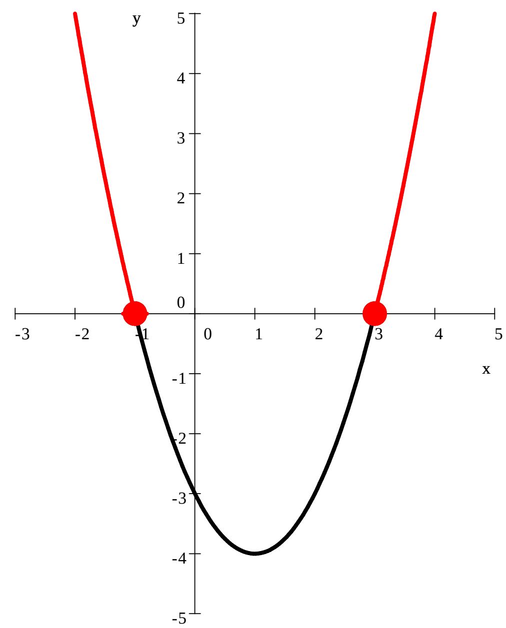
2. Solve the inequality $2x+3 \le x^2$.

Solution: We reduce one side to zero first and then factor. (There are several factoring techniques possible, we will complete the square.)

$$\begin{aligned} 2x+3 &\le x^2 \\ 0 &\le x^2-2x-3 \qquad\qquad (x-1)^2 = x^2-2x+1 \\ 0 &\le \underbrace{x^2-2x+1}-1-3 \\ 0 &\le (x-1)^2-4 \\ 0 &\le (x-1+2)(x-1-2) \\ 0 &\le (x+1)(x-3) \end{aligned}$$

The right-hand side is a quadratic expression with a positive leading coefficient. The graph of such an expression is a regular (or upward opening) parabola, with $x-$intercepts at $x=-1$ and $x=3$.

We graph the parabola.$y=(x+1)(x-3)$.
The inequality to be solved is $(x+1)(x-3) \ge 0$. For every point of the parabola, $y=(x+1)(x-3)$. So the inequality $(x+1)(x-3) \ge 0$ becomes $y \ge 0$ on the parabola. This means that we need to find the $x-$values for which $y$ is positive or zero. In other words, the $x-$coordinates of all points on the parabola that lie on or above the $x-$axis. These points are (on the parabola) to the left of $-1$ and to the right of 3.
The $x-$coordinate of these points range from $-\infty$ to $-11$ or from 3 to $\infty$. Thus the solution is $x \le -11$ or $x \ge 3$ The same result in interval notation: $\boxed{(-\infty,-11] \cup [3,\infty)}$.




$y=(x+1)(x-3)$

3. Solve the inequality $x^2-10x+20 \ge -9$.

Solution: We reduce one side to zero first and then factor.

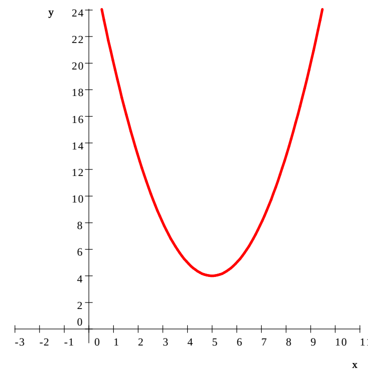
$$\begin{aligned} x^2-10x+20 &\ge -9 \\ x^2-10x+29 &\ge 0 \qquad\qquad (x-5)^2 = x^2-10x+29 \\ \underbrace{x^2-10x+25}-25+29 &\ge 0 \\ (x-5)^2+4 &\ge 0 \end{aligned}$$

The right-hand side is a quadratic expression that cannot be factored. This means that the equation $x^2-10x+29=0$ has no real solution. However, in case of inequalities, this is NOT the end of the story.

The right-hand side is a quadratic expression with a positive leading coefficient. The graph of such an expression is a regular (or upward opening) parabola, with vertex at $(5,4)$.

We plot the graph of $y = (x-5)^2 + 4$. Consider now the inequality to be solved: $(x-5)^2 + 4 \geq 0$. We need to find the $x-$values for which $y = (x-5)^2 + 4$ is positive or zero. In other words, the $x-$coordinates of all points on the parabola that lie on or above the $x-$axis. This is clearly true for every point on the parabola. Thus the solution set is the set of all real numbers: $\mathbb{R}$, or in interval notation: $\boxed{(-\infty, \infty)}$.




$y = (x-5)^2 + 4$

4. Solve the inequality $6x - x^2 \geq 0$

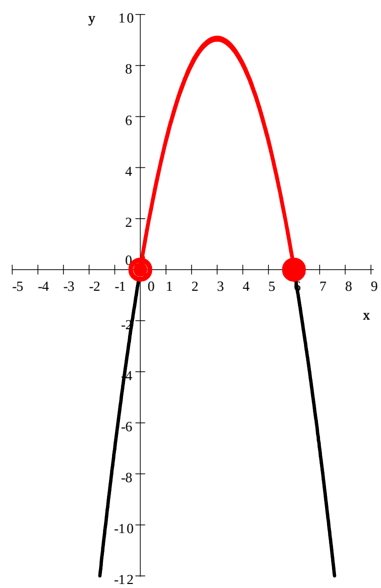
Solution: One side of the inequality is already zero. We just need to factor the left-hand side.

$$-x^2 + 6x \geq 0$$
$$-x(x-6) \geq 0$$

The left-hand side is a quadratic expression with a negative leading coefficient.

The graph of such an expression is an upside down (or downward opening) parabola, with $x-$intercepts at $x = 0$ and $x = 6$. We graph this parabola.

The inequality to be solved: $-x^2 + 6x \geq 0$. That is the same as $y \geq 0$ on the graph of $y = -x^2 + 6x$. Therefore, we must find all points on the parabola whose $y-$coordinate is positive or zero. Visually, those are the points of the graph that are on or above the $x-$axis. These points are between the zeroes, 0 and 6. Thus the solution is: $0 \leq x \leq 6$. The same result in interval notation is $\boxed{[0, 6]}$




$y = -x^2 + 6x$

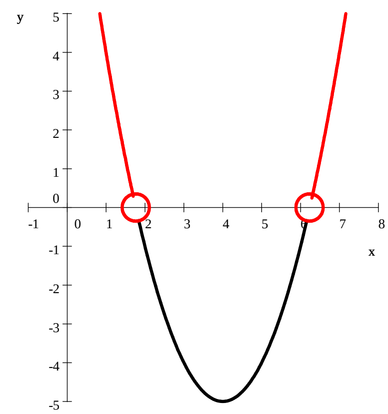
5. Solve the inequality $x^2 < 8x - 11$.

Solution: This problem is not trickier than the ones we have seen, only the numbers appearing are a bit more difficult. We reduce one side to zero first and then factor.

$$\begin{aligned} x^2 &< 8x - 11 \\ x^2 - 8x + 11 &< 0. \qquad (x-4)^2 = x^2 - 8x + 16 \\ 0 &< \underbrace{x^2 - 8x + 16} - 16 + 11 \\ 0 &< (x-4)^2 - 5 \\ 0 &< (x-4)^2 - \left(\sqrt{5}\right)^2 \\ 0 &< \left(x - 4 + \sqrt{5}\right)\left(x - 4 - \sqrt{5}\right) \end{aligned}$$

The right-hand side is a quadratic expression with a positive leading coefficient. The graph of such an expression is a regular (or upward opening) parabola, with $x-$intercepts at $x = 4 - \sqrt{5}$ and $x = 4 + \sqrt{5}$.

Recall that $\sqrt{5}$ is a number between 2 and 3. We can then place both of these zeroes on the positive part of the $x-$axis. We graph the parabola $y = \left(x - 4 + \sqrt{5}\right)\left(x - 4 - \sqrt{5}\right)$
The inequality to be solved is $\left(x - 4 + \sqrt{5}\right)\left(x - 4 - \sqrt{5}\right) \geq 0$. For every point of the parabola, $y = \left(x - 4 + \sqrt{5}\right)\left(x - 4 - \sqrt{5}\right)$ So the inequality $\left(x - 4 + \sqrt{5}\right)\left(x - 4 - \sqrt{5}\right) \geq 0$ becomes $y \geq 0$ on the graph of the parabola. This means that we need to find the $x-$values for which $y$ is positive. In other words, the $x-$coordinates of all points on the parabola that lie above the $x-$axis. These points are (on the parabola) to the left of $4 - \sqrt{5}$ and to the right of $4 + \sqrt{5}$.
The $x-$coordinates of these points range from $-\infty$ to $4 - \sqrt{5}$ or from $4 + \sqrt{5}$ to $\infty$.




$y = (x+1)(x-3)$

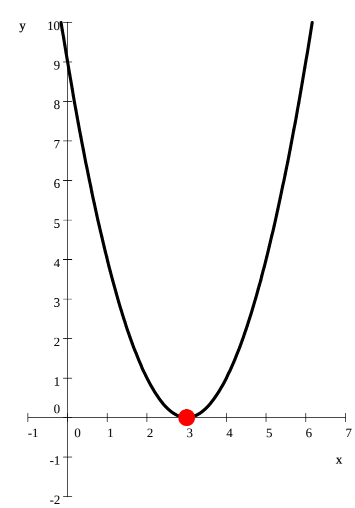
Thus the solution is: $x < 4 - \sqrt{5}$ or $x \geq 4 + \sqrt{5}$. The same result in interval notation is $\boxed{\left(-\infty, 4 - \sqrt{5}\right] \cup \left[4 + \sqrt{5}, \infty\right)}$

6. Solve the inequality $\frac{1}{2}x^2 - 3x + 5 \leq \frac{1}{2}$

Solution: As before, we first reduce one size to zero and factor the other side.

$$\begin{aligned}
\frac{1}{2}x^2 - 3x + 5 &\leq \frac{1}{2} \\
\frac{1}{2}x^2 - 3x + \frac{9}{2} &\leq 0 \qquad \text{multipy by 2} \\
x^2 - 6x + 9 &\leq 0 \qquad (x-3)^2 = x^2 - 6x + 9 \\
(x-3)^2 &\leq 0
\end{aligned}$$

The graph of $y = (x-3)^2$ is a regular (or upward opening) parabola, with vertex and $x-$intercept at $x = 3$. We graph this parabola.
The inequality to be solved is $(x-3)^2 \leq 0$. For every point on the parabola, $y = (x-3)^2$. So the inequality $(x-3)^2 \leq 0$ becomes $y \leq 0$ on the parabola. This means that we need to find the $x-$values for which $y$ is negative or zero. In other words, the $x-$coordinates of all points on the parabola that lie on or below the $x-$axis. Most of the parabola is above the $x$-axis, that is the same as squares are always non-egative. The only solution is $x = 3$, or in interval notation: $\boxed{[3, 3]}$.




$y = (x+1)(x-3)$

