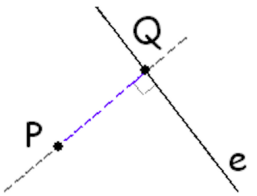
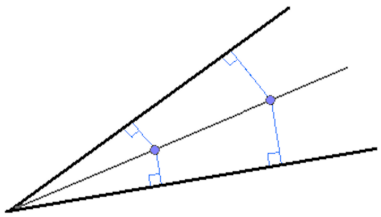
Theorem: For every triangle, there exists a circle that is located inside the triangle and all three sides of the triangle are tangents to the circle. We call such a circle the **inscribed circle** of the triangle.

We will prove this property. First we need a few definitions and theorems.

Definition: The distance between a point P and a line e is the distance between P and Q, where Q is the intersection of e and the line that is perpendicular to e and contains P.

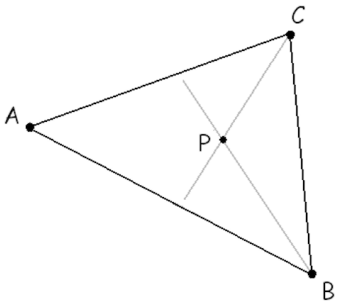


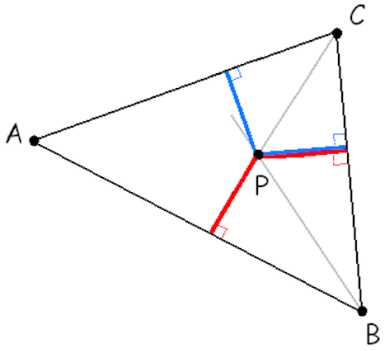
Definition: Given an angle, its bisector is the set of all points equidistant to both rays forming the angle.



Claim: The angle bisectors in a triangle intersect each other in a single point.

Proof: Let us draw the angle bisectors at vertices B and C, and let P be the point where the two angle bisectors intersect each other.



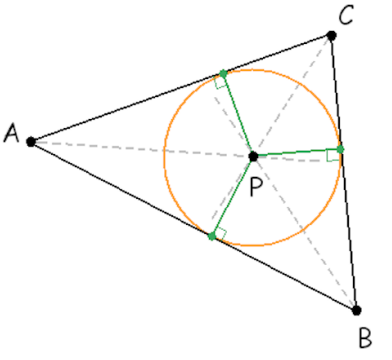


Because P is on the bisector of angle B, P is equidistant to lines AB and BC. Because P is on the bisector of angle C, P is equidistant to lines BC and AC. Therefore, P is the same distance from all three lines. Therefore, P is a point equidistant to all three sides.

This has two immediate consequences. If we draw in the third bisector, it too will contain P and therefore, the three angle bisectors intersect each other in a single point. Point P is equidistant to all three sides, therefore it is also the center of a circle we can draw inside the triangle. ■

We usually denote the radii of the inscribed and circumscribed circles by r and R, respectively.

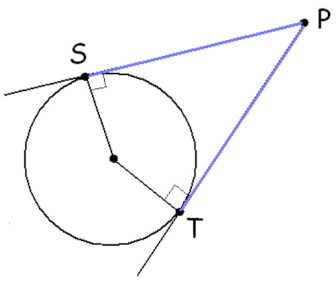
The following facts often come up in problems that involve the inscribed circle.



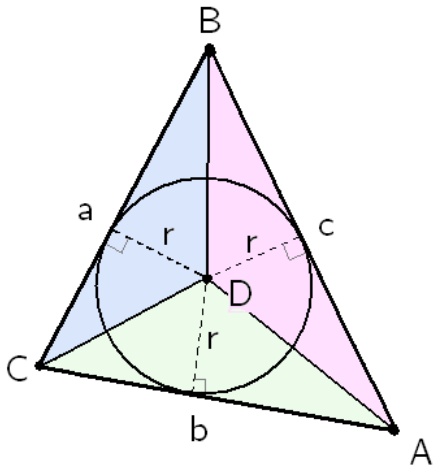
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Theorem: Suppose that P is a point outside of a circle and we draw the two tangent lines to the circle that contain P. Let T and S be the points of tangency. Then the line segments PT and PS are equally long.

Theorem: The radius drawn to the point of tangency is perpendicular to the tangent line.



Theorem: If P represents the perimeter of a triangle and r the radius of the inscribed circle, then the area of the triangle is $A = \frac{1}{2}Pr$.



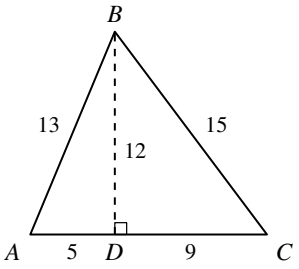
Proof: The area of triangle ABD is $A_1 = \frac{cr}{2}$. The area of triangle BDC is $A_2 = \frac{ar}{2}$. The area of triangle ADC is $A_3 = \frac{br}{2}$. The area of the entire triangle is

$$A = A_1 + A_2 + A_3 = \frac{cr}{2} + \frac{ar}{2} + \frac{br}{2} = \frac{r}{2}(a + b + c) = \frac{1}{2}rP$$

This concludes our proof. ■

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Example 1. The sides of a triangle are 13, 14, and 15 units long, and the height belonging to the 14 units long side is 12 units long. Find the radius of the inscribed circle.



Solution: The height is 12 units, so the area of the triangle is $A = \frac{1}{2} \cdot 14 \cdot 12 = 84$ unit^2. The perimeter is $P = 13 + 14 + 15 = 42$ units. Now we use the area formula $A = \frac{1}{2}Pr$ and solve for r:

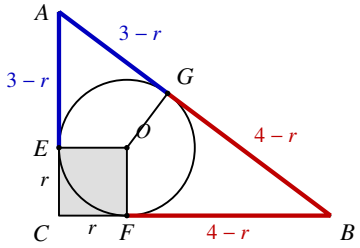
$$84 = \frac{1}{2} \cdot 42 \cdot r \qquad \Longrightarrow \qquad 84 = 21r \qquad \Longrightarrow \qquad r = 4$$

So the radius of the inscribed circle is 4 units long.

Remark: This triangle is made of two right triangles glued together along the height: the 5-12-13 and the 9-12-15 triangles. Gluing right triangles together is an easy way to make triangles whose sides and height are all whole numbers.

Example 2. Find the radius of the circle inscribed in the right triangle with legs 3 and 4 units long and hypotenuse 5 units long.

Solution 1: Let C be the vertex of the right angle, O the center of the inscribed circle, and E, F, G the points where the circle touches the sides CA, CB, and AB. The radius drawn to the point of tangency is perpendicular to the tangent line, so the angles at E and F are right angles. The angle at C is also a right angle, so $CEOF$ is a rectangle, and since $OE = OF = r$, it is a square. This is the little square in the corner: $CE = CF = r$.



So the rest of the legs are $AE = 3 - r$ and $BF = 4 - r$. By the theorem about the equally long tangent line segments drawn from a point, $AG = AE = 3 - r$ and $BG = BF = 4 - r$. The hypotenuse is made of these two segments:

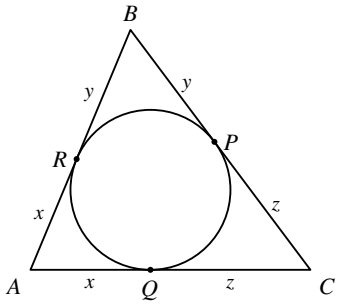
$$5 = (3 - r) + (4 - r) \qquad \Longrightarrow \qquad 5 = 7 - 2r \qquad \Longrightarrow \qquad 2r = 2 \qquad \Longrightarrow \qquad r = 1$$

Solution 2: In a right triangle the two legs are a base and the height belonging to it, so the area is $A = \frac{1}{2} \cdot 3 \cdot 4 = 6$ unit^2. The perimeter is $P = 3 + 4 + 5 = 12$ units, so $r = \frac{2A}{P} = \frac{2 \cdot 6}{12} = 1$. Both solutions give that the radius of the inscribed circle is 1 unit long.

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Example 3. The sides of triangle ABC are $AB = 13$, $BC = 15$, and $CA = 14$ units long. The inscribed circle touches the sides at points P, Q, and R, and these points split the sides into six line segments. Find the lengths of these line segments.

Solution: By the theorem about the equally long tangent line segments drawn from a point, the two segments starting at A are equally long; let us call their length x. Similarly, let y be the length of the two segments starting at B, and z the length of the two segments starting at C. Each side is made of two of these segments:



$$x + y = 13 \qquad y + z = 15 \qquad z + x = 14$$

If we add the three equations, every unknown appears twice: $2x + 2y + 2z = 42$, so $x + y + z = 21$. (This is half of the perimeter.) Now we subtract each equation from this one:

$$x = 21 - 15 = 6 \qquad y = 21 - 14 = 7 \qquad z = 21 - 13 = 8$$

So the segments starting at A are 6 units long, the ones starting at B are 7 units long, and the ones starting at C are 8 units long. In general: the tangent line segments starting at a vertex are as long as half the perimeter minus the side opposite to that vertex.

Example 4. The perimeter of a triangle is 30 units, and the radius of its inscribed circle is 2 units. Find the area of the triangle.

Solution: This time we use the formula $A = \frac{1}{2}Pr$ the other way around: we know P and r, and we compute A.

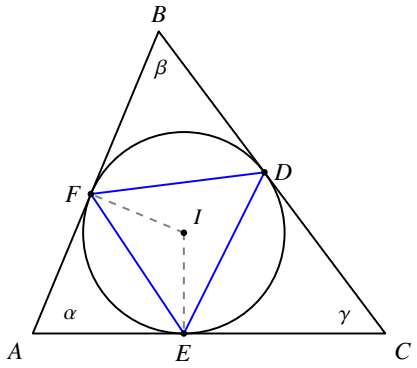
$$A = \frac{1}{2}Pr = \frac{1}{2} \cdot 30 \cdot 2 = 30$$

So the area of the triangle is 30 unit^2. (For example, the right triangle with legs 5 and 12 units long and hypotenuse 13 units long is such a triangle: its area is $\frac{1}{2} \cdot 5 \cdot 12 = 30$.)

Similarly, if we know the area and the radius, we can find the perimeter: $P = \frac{2A}{r}$.

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Example 5. The angles of triangle ABC are α at A, β at B, and γ at C. The inscribed circle touches the sides BC, CA, and AB at the points D, E, and F. Find the angles of triangle DEF.



Solution 1: By the theorem about the equally long tangent line segments drawn from a point, $BD = BF$. So triangle BDF is isosceles, and its angle at B is β. The two base angles of an isosceles triangle are equal, so

$$\angle BDF = \angle BFD = \frac{180^\circ - \beta}{2} = 90^\circ - \frac{\beta}{2}$$

Similarly, $CD = CE$, so $\angle CDE = 90^\circ - \frac{\gamma}{2}$. The three angles at D along the line BC add up to 180°:

$$\angle FDE = 180^\circ - \left(90^\circ - \frac{\beta}{2}\right) - \left(90^\circ - \frac{\gamma}{2}\right) = \frac{\beta + \gamma}{2} = \frac{180^\circ - \alpha}{2} = 90^\circ - \frac{\alpha}{2}$$

In the same way, the angle at E is $90^\circ - \frac{\beta}{2}$ and the angle at F is $90^\circ - \frac{\gamma}{2}$.

Solution 2: Let I be the center of the inscribed circle. The radius drawn to the point of tangency is perpendicular to the tangent line, so the quadrilateral $AEIF$ has right angles at E and at F. Its angles add up to 360°, so the central angle is $\angle EIF = 360^\circ - 90^\circ - 90^\circ - \alpha = 180^\circ - \alpha$. The point D is also on the circle (on the arc that is not between E and F), so $\angle EDF$ is an inscribed angle that belongs to the same arc EF. An inscribed angle is half of the central angle:

$$\angle EDF = \frac{1}{2}\angle EIF = \frac{180^\circ - \alpha}{2} = 90^\circ - \frac{\alpha}{2}$$

For example, if the angles of triangle ABC are 50°, 60°, and 70°, then the angles of triangle DEF are 65°, 60°, and 55°. Notice that every angle of triangle DEF is less than 90°: the triangle formed by the points of tangency is always an acute triangle, even if triangle ABC is obtuse.

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Practice Problems

1. Suppose that ABC is an isosceles triangle in which

   a) two angles are $50°$. What is the angle formed by the angle bisectors of these angles?

   b) two angles are α. What is the angle formed by the angle bisectors of these angles? Give your answer in terms of α.

2. Find the exact value of the length of the radius of the circle inscribed in a triangle with sides 14, 25, and 25 units long.

3. In case of an equilateral (regular) triangle, the center of the inscribed circle is the same as the center of the circumscribed circle. We sometimes call this point the center of the triangle itself. In the equilateral triangle ABC with all sides 18 units long, what is the distance between the center and point A?

4. The sides of a triangle are 15 units, 19 units, and 28 units long. Let P, Q, and R be the points where the inscribed circle touches the sides of the triangle. These points split each side into two line segments. Find the length of these line segments.

5. Suppose that a triangle has sides $AB = 10$, $BC = 15$, and $AC = 20$ units long. Let β be the angle opposite side AC. We draw the angle bisector of β. Let P be the intersection of this angle bisector and side AC. How long are the line segments AP and PC?

6. Suppose that the sides of an isosceles triangle are a, a, and b long. Let P denote the perimeter of the triangle. Prove that then $a + b < \frac{3}{4}P$.

7. Find the radius of the circle inscribed in a triangle with sides 13, 13, and 10 units long. The height belonging to the 10 units long side is 12 units long.

8. Find the radius of the circle inscribed in a triangle with sides 5, 5, and 6 units long. The height belonging to the 6 units long side is 4 units long.

9. Find the radius of the circle inscribed in a triangle with sides 10, 17, and 21 units long. The height belonging to the 21 units long side is 8 units long.

10. The legs of a right triangle are 6 and 8 units long, and its hypotenuse is 10 units long. Find the radius of the inscribed circle in two different ways: a) using the area formula $A = \frac{1}{2}Pr$, b) using the equally long tangent line segments.

11. Use the equally long tangent line segments to find the radius of the circle inscribed in the right triangle with legs 5 and 12 units long and hypotenuse 13 units long.

12. Use the equally long tangent line segments to find the radius of the circle inscribed in the right triangle with legs 8 and 15 units long and hypotenuse 17 units long.

13. The legs of a right triangle are 20 and 21 units long, and its hypotenuse is 29 units long. Find the radius of the inscribed circle.

14. The legs of a right triangle are a and b units long, and its hypotenuse is c units long. Prove that the radius of the inscribed circle is $r = \frac{a + b - c}{2}$.

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15. The area of a triangle is 54 unit^2, and the radius of its inscribed circle is 3 units. Find the perimeter of the triangle.

16. The perimeter of a triangle is 42 units, and the radius of its inscribed circle is 4 units. Find the area of the triangle.

17. (Enrichment) The formula $A = \frac{1}{2}Pr$ is true not only for triangles, but for every polygon that has an inscribed circle (a circle that touches all of its sides). The proof is the same: connect the center of the circle to the vertices, and add up the areas of the triangles. Check the formula for a square with sides a units long. (What is the radius of the inscribed circle?)

18. (Enrichment) A rhombus has sides 10 units long, and its height is 8 units. Use the formula $A = \frac{1}{2}Pr$ to find the radius of the circle inscribed in the rhombus. Compare your answer with the height.

19. The angles of a triangle are 40°, 60°, and 80°. The inscribed circle touches the sides at three points. Find the angles of the triangle formed by these three points.

20. The angles of a triangle are 20°, 30°, and 130°. Find the angles of the triangle formed by the three points where the inscribed circle touches the sides. Is this new triangle obtuse too?

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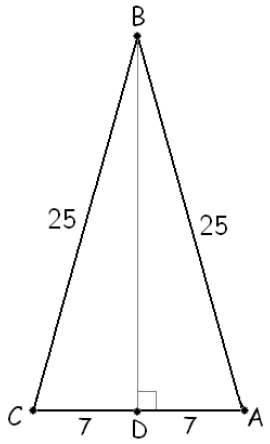
Answers

1. a) 50° b) α 2. $\frac{21}{4} = 5.25$ 3. $6\sqrt{3}$ 4. 3, 12, 16 units 5. 8 and 12 units 6. see solutions
7. $\frac{10}{3}$ 8. $\frac{3}{2}$ 9. $\frac{7}{2}$ 10. 2 11. 2 12. 3 13. 6 14. see solutions
15. 36 units 16. 84 unit^2 17. see solutions 18. 4 units 19. $70^\circ, 60^\circ, 50^\circ$ 20. $80^\circ, 75^\circ, 25^\circ$; acute

Hints, Solutions

1. Let P be the intersection point of the bisectors of the equal angles, at points A and B. What are the angles in triangle ABP?

2. Find the exact value of the length of the radius of the circle inscribed in a triangle with sides 14, 25, and 25 units long.



Solution 1: Recall that there is an area formula for the triangle that includes the radius of the inscribed circle, $A = \frac{1}{2}Pr$. We state this for the triangle and solve for r.

For the area, we need to compute the height (BD) of the triangle. We use the Pythagorean theorem.

$$h = \sqrt{25^2 - 7^2} = \sqrt{576} = 24$$

Then the area is $\frac{1}{2} \cdot 24 \cdot 14 = 168$ unit^2, and the perimeter is $P = 25 + 25 + 14 = 64$ units. We are now ready to solve for r.

$$A = \frac{1}{2}Pr \qquad \Rightarrow \qquad r = \frac{2A}{P} = \frac{2 \cdot 168}{64} = \frac{21}{4} = 5.25$$

Solution 2: We compute the height (BD) of the triangle using the Pythagorean theorem.

$$h = \sqrt{25^2 - 7^2} = \sqrt{576} = 24$$

Line segments AD and AQ are of the same length because they are both tangents drawn to the inscribed circle. Therefore, $AQ = 7$ units. Now we know that QB is 18 units long. BP is $24 - r$ units long. We can find r by stating the Pythagorean theorem for triangle BPQ.

$$\begin{aligned} r^2 + 18^2 &= (24 - r)^2 \\ r^2 + 18^2 &= r^2 - 48r + 24^2 \\ 48r &= 252 \\ r &= \frac{252}{48} = \frac{21}{4} = 5.25 \end{aligned}$$

So the radius is 5.25 units long.

3. Hint: there are several ways to solve this. The center O is on the height drawn from A to side BC, and it separates this height into R and r. First use the Pythagorean theorem to find the height: $h = \sqrt{18^2 - 9^2} = \sqrt{243} = 9\sqrt{3}$. Then use $A = \frac{1}{2}rP$ to find r (the area is $A = \frac{1}{2} \cdot 18 \cdot 9\sqrt{3} = 81\sqrt{3}$ and the perimeter is 54, so $r = 3\sqrt{3}$), and then subtract it from the height: $9\sqrt{3} - 3\sqrt{3} = 6\sqrt{3}$.

4. Hint: the tangent line segments from an external point are equally long. Use this to set up and then solve a system of 3 linear equations.

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5. Hint: Consider triangles APB and BPC. Compare the areas (as a ratio) of these two triangles. We can compute the areas of these two triangles in two different ways, so we can find the ratio of these triangles.

6. **Solution:** Let us first investigate what this statement would mean if it was true. We first substitute $P = 2a+b$ into the inequality and simplify:

$$\begin{aligned} a + b &< \frac{3}{4}P \\ a + b &< \frac{3}{4}(2a + b) \\ a + b &< \frac{3}{2}a + \frac{3}{4}b \qquad \text{subtract } a \text{ and } \frac{3}{4}b \\ b - \frac{3}{4}b &< \frac{3}{2}a - a \\ \frac{b}{4} &< \frac{a}{2} \qquad \text{multiply by 2} \\ \frac{b}{2} &< a \qquad \text{or} \qquad b < 2a \end{aligned}$$

   There are two ways of continuing this problem, both correct.

   Solution 1. If a triangle has sides a, a, and b, then the triangle inequality $a+a > b$ must be true. We can present the computation above backwards, starting with $b < 2a$ to $a + b < \frac{3}{4}P$. In other words, from $b < 2a$ it follows that $a + b < \frac{3}{4}P$.

   Solution 2. If we draw in the height belonging to the base b, this line splits the triangle into two right triangles. Considering either one, we see that a is the hypotenuse of the right triangle, and another shorter side is $\frac{b}{2}$. The hypotenuse is always the longest side in the triangle, so $\frac{b}{2} < a$ must be true. We can present the computation above backwards, starting with $\frac{b}{2} < a$ to $a + b < \frac{3}{4}P$. In other words, from $\frac{b}{2} < a$ it follows that $a + b < \frac{3}{4}P$.

7. The area is $A = \frac{1}{2} \cdot 10 \cdot 12 = 60$ and the perimeter is $P = 36$, so $r = \frac{2A}{P} = \frac{120}{36} = \frac{10}{3}$.

8. The area is $A = \frac{1}{2} \cdot 6 \cdot 4 = 12$ and the perimeter is $P = 16$, so $r = \frac{2A}{P} = \frac{24}{16} = \frac{3}{2}$.

9. The area is $A = \frac{1}{2} \cdot 21 \cdot 8 = 84$ and the perimeter is $P = 48$, so $r = \frac{2A}{P} = \frac{168}{48} = \frac{7}{2}$. (This triangle is made of the right triangles 6-8-10 and 15-8-17, glued together along the height 8.)

10. a) The legs are a base and its height, so $A = \frac{1}{2} \cdot 6 \cdot 8 = 24$. The perimeter is $P = 24$, so $r = \frac{2 \cdot 24}{24} = 2$. b) The corner square gives the tangent segments $6 - r$ and $8 - r$ on the hypotenuse, so $10 = (6 - r) + (8 - r)$, and $r = 2$.

11. $13 = (5 - r) + (12 - r)$, so $2r = 4$ and $r = 2$.

12. $17 = (8 - r) + (15 - r)$, so $2r = 6$ and $r = 3$.

13. Either way: $29 = (20 - r) + (21 - r)$ gives $r = 6$; or $A = \frac{1}{2} \cdot 20 \cdot 21 = 210$, $P = 70$, and $r = \frac{2 \cdot 210}{70} = 6$.

14. **Solution:** As in the example with the right triangle with legs 3 and 4 units long, the radii drawn to the legs and the legs themselves form a little square of side r in the corner of the right angle. So the tangent line segments from the other two vertices are $a - r$ and $b - r$, and the hypotenuse is made of these two segments: $c = (a - r) + (b - r)$. Solving for r: $2r = a + b - c$, so $r = \frac{a + b - c}{2}$.

15. $54 = \frac{1}{2} \cdot P \cdot 3$, so $P = \frac{2 \cdot 54}{3} = 36$ units.

16. $A = \frac{1}{2} \cdot 42 \cdot 4 = 84$ unit^2.

17. The inscribed circle of a square touches two opposite sides, so its diameter is a and its radius is $r = \frac{a}{2}$. The perimeter is $P = 4a$, so $\frac{1}{2}Pr = \frac{1}{2} \cdot 4a \cdot \frac{a}{2} = a^2$, which is indeed the area of the square.

18. The area of the rhombus is $A = 10 \cdot 8 = 80$ unit^2 (side times height), and its perimeter is $P = 40$ units. So $r = \frac{2A}{P} = \frac{160}{40} = 4$ units. This is half of the height: the inscribed circle touches two opposite sides, so its diameter is the height of the rhombus.

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19. The angle at the point of tangency opposite a vertex is 90° minus half of the angle at that vertex: $90^\circ - 20^\circ = 70^\circ$, $90^\circ - 30^\circ = 60^\circ$, and $90^\circ - 40^\circ = 50^\circ$.

20. $90^\circ - 10^\circ = 80^\circ$, $90^\circ - 15^\circ = 75^\circ$, and $90^\circ - 65^\circ = 25^\circ$. The new triangle is acute: every angle of the form $90^\circ - \frac{\alpha}{2}$ is less than 90°.

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