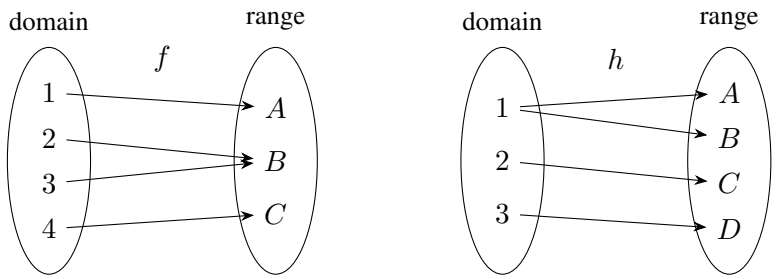
Recall the following definitions of relations and functions.

Definition: A **relation** is an assignment between elements of a non-empty set called the domain and another set called the range. We usually denote relations by lower case letters such as f or g.

Definition: A **function** is an assignment between elements of a non-empty set called the domain and another set called the range, with one restriction: that to each element of the domain, only one thing can be assigned. We usually denote functions by lower case letters such as f or g.

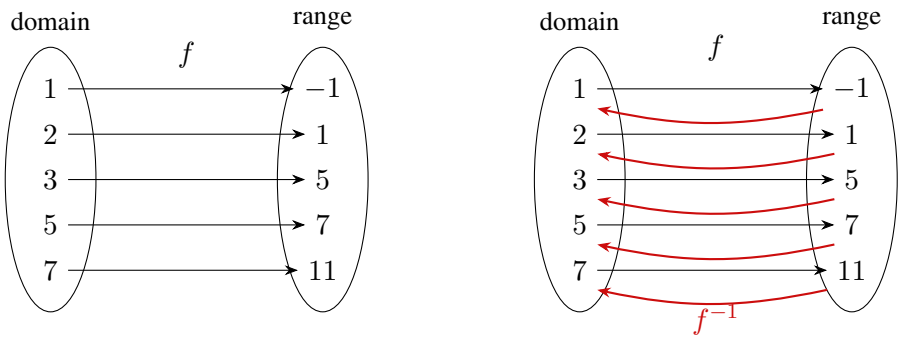


The picture above shows two relations, f and h. While f is a function, h is not, because more than one value is assigned to 1 under h. The distinction between functions and relations that are not functions will be important later.

The main idea about the inverse is the intention of traveling backward along the assignments.

Definition: Suppose that f is a function. The **inverse relation**, denoted by f^{-1}, is an assignment that is the reversal of the assignment of f.

Consider the function f shown on the picture below. The inverse relation, denoted by f^{-1}, is obtained by reversing the assignments defined by f.



So the statement $f(1) = -1$ can be re-phrased as $f^{-1}(-1) = 1$; the statement $f(2) = 1$ as $f^{-1}(1) = 2$; the statement $f(3) = 5$ as $f^{-1}(5) = 3$, and so on, $f^{-1}(7) = 5$ and $f^{-1}(11) = 7$.

Note 1: It is important to notice that the domain and range of f swap roles under the inverse relation: the domain of f is the range of f^{-1} and the range of f is the domain of f^{-1}. This fact will be useful later.

Note 2: The notation for inverse relation and reciprocal are similar but not quite the same. $f^{-1}(7)$ denotes the inverse

relation while $f(7)^{-1}$ denotes the reciprocal. So,

$$f^{-1}(7) = 5 \quad \text{and} \quad f(7)^{-1} = \frac{1}{f(7)} = \frac{1}{11}$$

Needless to say, it will be very important for us not to confuse the two notations.

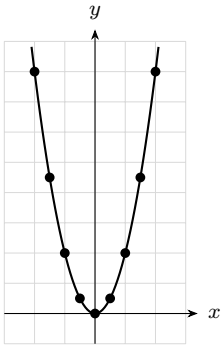
Given the graph of a function f, how do we graph the inverse relation? There are two methods to graph the inverse: by transposing coordinates on the graph of f and by reflecting the graph of f to the line $y = x$.

Example 1. Graph the inverse relation f^{-1} for the function $f(x) = x^2$.

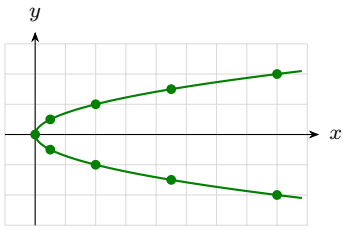
Solution: Let us graph $y = x^2$ first. Since the inverse literally reverses the assignment of f, x becomes y and y becomes x under the inverse relation. So if $(3, 9)$ and $(-3, 9)$ are points on the graph of f, then switching the two coordinates gives us points on the graph of the inverse. So, $(9, 3)$ and $(9, -3)$ are points on the graph of the inverse. We apply this method to more points:

points on f	$(0,0)$	$(1,1)$	$(-1,1)$	$(2,4)$	$(-2,4)$	$(3,9)$	$(-3,9)$	$(4,16)$	$(-4,16)$
points on f^{-1}	$(0,0)$	$(1,1)$	$(1,-1)$	$(4,2)$	$(4,-2)$	$(9,3)$	$(9,-3)$	$(16,4)$	$(16,-4)$

We now plot these points and connect them.

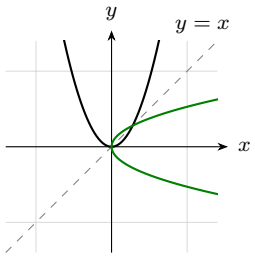


graph of f



graph of f^{-1}

We can graph these two together to better see the symmetry.

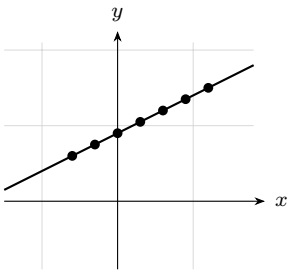


Although it appears that this is a rotation by $90°$, this is not what happens, because the point $(3, 9)$ becomes $(9, 3)$ on the inverse graph and the point $(-3, 9)$ becomes $(9, -3)$. The fact that this is generally not a rotation will be more clear after the next example.

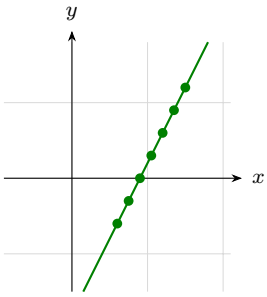
Example 2. Graph the inverse relation g^{-1} for the function $g(x) = \frac{1}{2}x + 3$.

Solution: We will again collect a few points on the graph of g and transpose their coordinates to obtain points on the graph of g^{-1}.

points on g	$(-2, 2)$	$\left(-1, \frac{5}{2}\right)$	$(0, 3)$	$\left(1, \frac{7}{2}\right)$	$(2, 4)$	$\left(3, \frac{9}{2}\right)$	$(4, 5)$
points on g^{-1}	$(2, -2)$	$\left(\frac{5}{2}, -1\right)$	$(3, 0)$	$\left(\frac{7}{2}, 1\right)$	$(4, 2)$	$\left(\frac{9}{2}, 3\right)$	$(5, 4)$

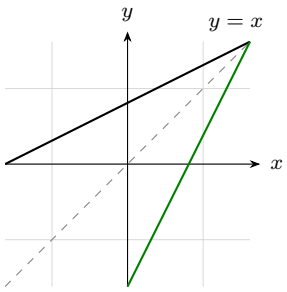


graph of g



graph of g^{-1}

We can graph these two together to better see the symmetry:



Notice that g^{-1} turned out to be a function but not f^{-1}. Since $f(3) = f(-3) = 9$, the inverse relation f^{-1} will fail to be a function because 9 will be assigned to both 3 and -3 under f^{-1}. Indeed, if a function assigns the same y-value to two different x-values, then the inverse relation will fail to be a function.

Theorem: Suppose that f is a function. The **inverse relation**, denoted by f^{-1}, is a function if and only if the function is one-to-one.

Recall that a function f is one-to-one if to each element of its domain a different element of the range is assigned.

Definition: A function f is one-to-one (or injective) if for all a and b in its domain, if $a \neq b$, then $f(a) \neq f(b)$.

As it turns out, inverse functions are extremely important for mathematicians. Even if a function is not one-to-one, we would still like to define an inverse *function* (and not just a relation): we restrict its domain until it becomes one-to-one and then take the inverse.

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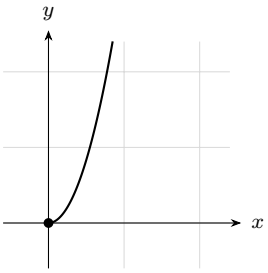
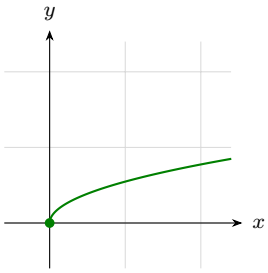
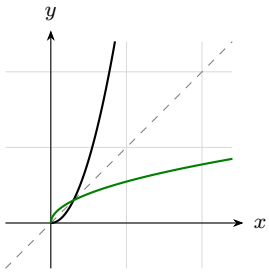
For example, $f(x) = x^2$ is not one-to-one on its full domain, the set of all real numbers. So we restrict its domain to $[0, \infty)$. On this smaller domain, the function is now one-to-one and so the inverse relation is a function.



$f(x) = x^2,\ x \geq 0$



graph of f^{-1}



graphs of f and f^{-1}

This inverse function is the square root function, $f^{-1}(x) = \sqrt{x}$.

Given a function f, how do we compute the formula for the inverse? While f gives an assignment from the x-values to the y-values, the inverse establishes an assignment from y back to x. The following is the algorithm to find the formula for the inverse function.

Step 1. Remove the function notation ($f(x)$) and replace it with y.

Step 2. Solve for x in the equation for f.

Step 3. To switch to the inverse, swap x and y in the equation.

Step 4. Remove y and replace it with function notation ($f^{-1}(x)$).

Example 3. Find the equation for the inverse of $f(x) = 2\sqrt[3]{x+1} - 3$.

Solution: We will write first $y = 2\sqrt[3]{x+1} - 3$. Then we solve for x in terms of y.

$$\begin{aligned} y &= 2\sqrt[3]{x+1} - 3 && \text{add 3} \\ y+3 &= 2\sqrt[3]{x+1} && \text{divide by 2} \\ \frac{y+3}{2} &= \sqrt[3]{x+1} && \text{raise to the third power} \\ \left(\frac{y+3}{2}\right)^3 &= x+1 && \text{subtract 1} \\ \left(\frac{y+3}{2}\right)^3 - 1 &= x \end{aligned}$$

The assignment of function f is $x = \left(\frac{y+3}{2}\right)^3 - 1$. Under the inverse, x and y are now swapped.

$$\boxed{x = \left(\frac{y+3}{2}\right)^3 - 1 \quad \text{under } f} \implies \boxed{y = \left(\frac{x+3}{2}\right)^3 - 1 \quad \text{under the inverse, } f^{-1}}$$

and so the inverse of f is $f^{-1}(x) = \left(\frac{x+3}{2}\right)^3 - 1$.

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Note: if we say 'the inverse of f', without specifying whether it is the inverse function or the inverse relation, we always mean the inverse function.

There is one more important and useful property of inverse functions.

Example 4. Suppose that $f(x) = 3x - 8$.

a) Find the inverse relation of f. b) Is f^{-1} a function?

c) Compute $f\left(f^{-1}(7)\right)$. d) Compute $f^{-1}(f(7))$.

Solution: a) As before, we drop the function notation, solve for x, and swap x and y.

$$\begin{aligned} y &= 3x - 8 && \text{add } 8 \\ y + 8 &= 3x && \text{divide by } 3 \\ \frac{y+8}{3} &= x && \text{swap } x \text{ and } y\text{: in the inverse, } y = \frac{x+8}{3} \end{aligned}$$

Therefore, $f^{-1}(x) = \dfrac{x+8}{3}$.

b) The formula $x = \dfrac{y+8}{3}$ shows that given y, the value of x is uniquely determined by f. In other words, no two different x-values 'get' the same y assigned. Therefore, $f^{-1}(x) = \dfrac{x+8}{3}$ is a function. In such a case, we say that f is one-to-one.

c) $f\left(f^{-1}(7)\right) = f\left(\dfrac{7+8}{3}\right) = f(5) = 3 \cdot 5 - 8 = 7$

d) $f^{-1}(f(7)) = f^{-1}(3 \cdot 7 - 8) = f^{-1}(13) = \dfrac{13+8}{3} = 7$

Theorem: Suppose that f is a one-to-one function. Then f^{-1} is a function such that for all x,

$$f^{-1}(f(x)) = x \quad \text{and} \quad f\left(f^{-1}(x)\right) = x$$

Another way to state this is: $f \circ f^{-1} = id$ and $f^{-1} \circ f = id$, where $id(x) = x$ for all x (the identity function).

Example 5. The table below shows some values of a one-to-one function f.

x	-2	-1	0	1	2	3
$f(x)$	7	4	1	-2	-5	-8

Find each of the following. a) $f(1)$ b) $f^{-1}(1)$ c) $f^{-1}(-8)$ d) $(f(1))^{-1}$

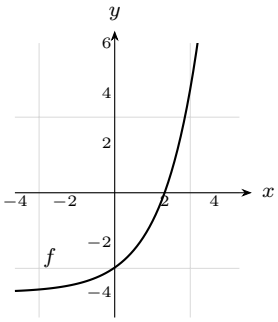
Solution: a) We look for 1 in the x row. Below it we find -2, so $f(1) = -2$.

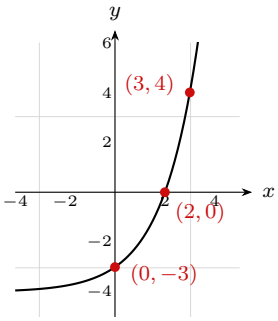
b) $f^{-1}(1)$ asks the question backward: which x is sent to 1 by f? We look for 1 in the $f(x)$ row. Above it we find 0, because $f(0) = 1$. So $f^{-1}(1) = 0$.

c) We look for -8 in the $f(x)$ row. Above it we find 3, because $f(3) = -8$. So $f^{-1}(-8) = 3$.

d) This one is not about the inverse at all: it is the reciprocal of $f(1)$. Since $f(1) = -2$, $(f(1))^{-1} = (-2)^{-1} = \dfrac{1}{-2} = -\dfrac{1}{2}$.

Example 6. The picture shows the graph of a one-to-one function f.
Use the graph to find each of the following.
a) $f(0)$ b) $f^{-1}(0)$ c) $f^{-1}(4)$





Solution: a) $f(0)$ is the y-coordinate of the point on the graph where $x = 0$. The graph crosses the y-axis at $(0, -3)$, so $f(0) = -3$.

b) $f^{-1}(0)$ is the x-value that f sends to 0. So we look for the point on the graph where $y = 0$. The graph crosses the x-axis at $(2, 0)$, so $f(2) = 0$ and $f^{-1}(0) = 2$.

c) We look for the point on the graph where $y = 4$. That point is $(3, 4)$, so $f(3) = 4$ and $f^{-1}(4) = 3$.

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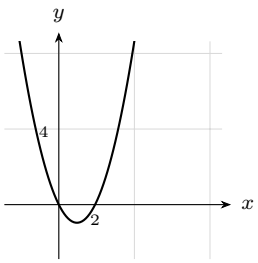
Example 7. The picture shows the graph of $f(x) = x(x-2)$. Graph the inverse relation in the same coordinate system.

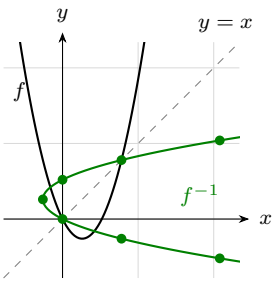


Solution: Let us find some points on the function. Substituting easy values for x, we obtain the following points on the graph of f.

$$(-2,8),\ (-1,3),\ (0,0),\ (1,-1),\ (2,0),\ (3,3),\ (4,8)$$

To obtain points on the graph of the inverse, we simply transpose the coordinates in each point. Thus the following are points on the graph of the inverse:

$$(8,-2),\ (3,-1),\ (0,0),\ (-1,1),\ (0,2),\ (3,3),\ (8,4)$$



We plot these points and connect them. It is also useful to keep in mind that the two graphs should be symmetrical to the line $y = x$.

Notice that the inverse relation is not a function: it assigns two values, 0 and 2, to the number 0. This is because f is not one-to-one, since $f(0) = f(2) = 0$.

Example 8. Find an equation for the inverse for each of the functions given.

a) $f(x) = 3x + 2$ b) $f(x) = (5x-1)^3$ c) $f(x) = \dfrac{x+4}{3x-5}$

d) $f(x) = \log_5(2x-1)$ e) $f(x) = e^{5x-1}$

Solution: In each case, first we drop the function notation and write y instead of $f(x)$. Then we solve for x and finally, swap x and y.

a) $f(x) = 3x + 2$

$$\begin{aligned} y &= 3x+2 && \text{subtract 2} \\ y-2 &= 3x && \text{divide by 3} \\ \frac{y-2}{3} &= x && \text{swap } x \text{ and } y\text{: } y = \frac{x-2}{3} \end{aligned}$$

So $f^{-1}(x) = \dfrac{x-2}{3}$.

b) $f(x) = (5x-1)^3$

$$\begin{aligned} y &= (5x-1)^3 && \text{take the third root of both sides} \\ \sqrt[3]{y} &= 5x-1 && \text{add 1} \\ \sqrt[3]{y}+1 &= 5x && \text{divide by 5} \\ \frac{\sqrt[3]{y}+1}{5} &= x && \text{swap } x \text{ and } y\text{: } y = \frac{\sqrt[3]{x}+1}{5} \end{aligned}$$

So $f^{-1}(x) = \dfrac{\sqrt[3]{x}+1}{5}$.

c) $f(x) = \dfrac{x+4}{3x-5}$

$$\begin{aligned} y &= \frac{x+4}{3x-5} && \text{multiply by } 3x-5 \\ y(3x-5) &= x+4 && \text{distribute} \\ 3xy-5y &= x+4 && \text{add } 5y\text{, subtract } x \\ 3xy-x &= 5y+4 && \text{factor out } x \\ x(3y-1) &= 5y+4 && \text{divide by } 3y-1 \\ x &= \frac{5y+4}{3y-1} && \text{swap } x \text{ and } y\text{: } y = \frac{5x+4}{3x-1} \end{aligned}$$

So $f^{-1}(x) = \dfrac{5x+4}{3x-1}$.

d) $f(x) = \log_5(2x-1)$

$$\begin{aligned} y &= \log_5(2x-1) && \text{re-write it as an exponential statement} \\ 5^y &= 2x-1 && \text{add 1} \\ 5^y+1 &= 2x && \text{divide by 2} \\ \frac{5^y+1}{2} &= x && \text{swap } x \text{ and } y\text{: } y = \frac{5^x+1}{2} \end{aligned}$$

So $f^{-1}(x) = \dfrac{5^x+1}{2}$.

e) $f(x) = e^{5x-1}$

$$\begin{aligned} y &= e^{5x-1} && \text{take the natural logarithm of both sides} \\ \ln y &= \ln\left(e^{5x-1}\right) && \ln\left(e^{5x-1}\right) = 5x-1 \\ \ln y &= 5x-1 && \text{add 1} \\ \ln y+1 &= 5x && \text{divide by 5} \\ \frac{\ln y+1}{5} &= x && \text{swap } x \text{ and } y\text{: } y = \frac{\ln x+1}{5} \end{aligned}$$

So $f^{-1}(x) = \dfrac{\ln x+1}{5}$.

Example 9. Prove that the inverse of a linear function is also linear and the two slopes are reciprocals of each other.

Solution: Let $f(x) = mx+b$. If $m = 0$, the function is constant, so it is very badly not one-to-one and the inverse function does not exist. If $m \neq 0$, then we can solve for x:

$$\begin{aligned} y &= mx+b && \text{subtract } b \\ y-b &= mx && \text{divide by } m \text{ (we can, since } m \neq 0) \\ \frac{y-b}{m} &= x && \text{swap } x \text{ and } y\text{: } y = \frac{x-b}{m} = \frac{1}{m}x - \frac{b}{m} \end{aligned}$$

So $f^{-1}(x) = \dfrac{1}{m}x - \dfrac{b}{m}$. This is a linear function and its slope is $\dfrac{1}{m}$, the reciprocal of m.

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Example 10. Let $f(x) = 5 - x^2$ with domain $x \geq 0$.

a) Find the inverse of f. b) State the domain and range of f and of f^{-1}.

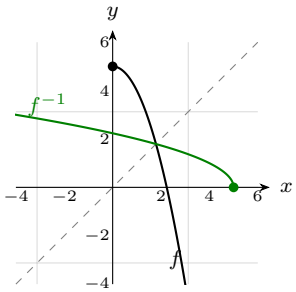
Solution: a) We drop the function notation and solve for x.

$$\begin{aligned} y &= 5 - x^2 && \text{add } x^2\text{, subtract } y \\ x^2 &= 5 - y && \text{take the square root of both sides} \\ x &= \pm\sqrt{5-y} \end{aligned}$$

This is where the restricted domain matters. Since $x \geq 0$, we keep only the non-negative square root: $x = \sqrt{5-y}$. Now we swap x and y: $y = \sqrt{5-x}$. So $f^{-1}(x) = \sqrt{5-x}$.

b) The domain of f is $[0, \infty)$, as given. When $x = 0$, $f(0) = 5$, and as x grows, $5 - x^2$ decreases without bound. So the range of f is $(-\infty, 5]$.

The domain and range swap roles under the inverse, so the domain of f^{-1} is $(-\infty, 5]$ and its range is $[0, \infty)$. This agrees with the formula: $\sqrt{5-x}$ is only defined when $5 - x \geq 0$, that is, when $x \leq 5$, and a square root is never negative.



Check: $f(2) = 5 - 4 = 1$ and $f^{-1}(1) = \sqrt{5-1} = \sqrt{4} = 2$. Indeed, f^{-1} takes us back from 1 to 2.

Example 11. Determine whether the given functions f and g are inverses of each other.

a) $f(x) = 2x^3 - 1$ and $g(x) = \sqrt[3]{\dfrac{x+1}{2}}$ b) $f(x) = x^3 + 2$ and $g(x) = \sqrt[3]{x} - 2$

Solution: Two functions are inverses of each other exactly when $f(g(x)) = x$ and $g(f(x)) = x$ for all x. So we compute both compositions.

a) $f(g(x)) = 2\left(\sqrt[3]{\dfrac{x+1}{2}}\right)^3 - 1 = 2 \cdot \dfrac{x+1}{2} - 1 = x + 1 - 1 = x$

$g(f(x)) = \sqrt[3]{\dfrac{(2x^3-1)+1}{2}} = \sqrt[3]{\dfrac{2x^3}{2}} = \sqrt[3]{x^3} = x$

Both compositions give x, so f and g are inverses of each other.

b) $f(g(x)) = (\sqrt[3]{x} - 2)^3 + 2$. This is not x. To see this clearly, it is enough to find one value where it fails. For example, if $x = 8$, then $g(8) = \sqrt[3]{8} - 2 = 0$ and $f(0) = 0^3 + 2 = 2$. So $f(g(8)) = 2 \neq 8$, and f and g are not inverses of each other.

(The inverse of f is actually $f^{-1}(x) = \sqrt[3]{x-2}$: the 2 has to be subtracted *before* taking the third root.)

Enrichment

Suppose that f is not a one-to-one function. So we restrict its domain until it does become one-to-one. Then we take the inverse function. One statement from $f^{-1}(f(x)) = x$ and $f\left(f^{-1}(x)\right) = x$ is still true. Which one? Give an example for how one of these two fails to be true.

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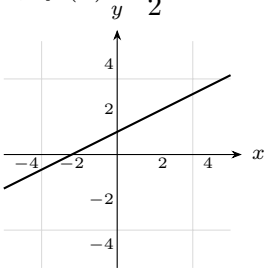
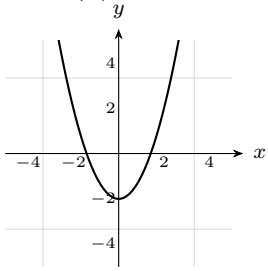
Practice Problems

1. The following pictures show the graphs of functions. In each case, graph the inverse relation.

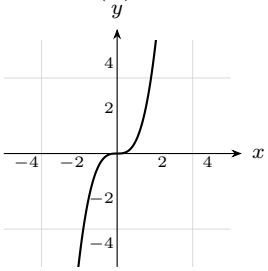
a) $f(x) = \frac{1}{2}x + 1$



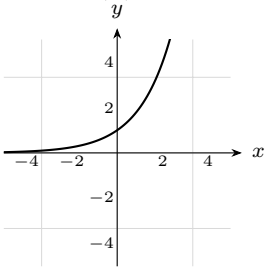
b) $f(x) = x^2 - 2$



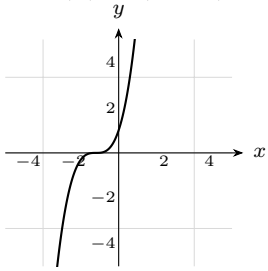
c) $f(x) = x^3$



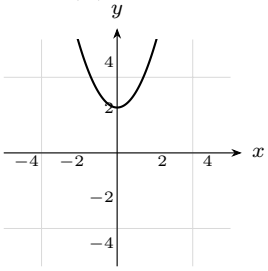
d) $f(x) = 2^x$



e) $f(x) = (x+1)^3$



f) $f(x) = x^2 + 2$



2. Find an equation for the inverse of each of the following functions.

a) $f(x) = 3^{5x-1}$

b) $f(x) = \dfrac{2x-7}{3x+5}$

c) $f(x) = \ln(2x-1)$

d) $f(x) = \dfrac{1}{3x+1}$

e) $f(x) = \dfrac{3\sqrt{x}-5}{2}$

f) $f(x) = \log_3\left(\frac{1}{2}x - 7\right)$

g) $f(x) = \dfrac{2x+5}{3x-2}$

h) $f(x) = 2 \cdot 5^x - 3$

i) $f(x) = \dfrac{4x-1}{x+2}$

3. The table below shows some values of a one-to-one function g.

x	-3	-1	0	2	5
$g(x)$	4	0	-2	7	3

Find each of the following. a) $g(2)$ b) $g^{-1}(0)$ c) $g^{-1}(3)$ d) $(g(5))^{-1}$ e) $g\left(g^{-1}(4)\right)$ f) $g^{-1}(g(-3))$

4. Determine whether the given functions f and g are inverses of each other.

a) $f(x) = 3x - 6,\ g(x) = \frac{1}{3}x + 2$ b) $f(x) = x^3 - 1,\ g(x) = \sqrt[3]{x} + 1$ c) $f(x) = \dfrac{x+1}{x-2},\ g(x) = \dfrac{2x+1}{x-1}$

5. Find the inverse of each function, and state the domain and range of f and of f^{-1}.

a) $f(x) = x^2 + 3,\ \ x \geq 0$ b) $f(x) = (x-2)^2,\ \ x \geq 2$

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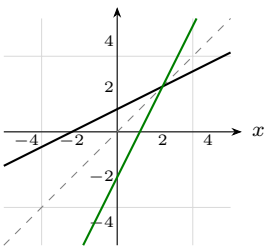
Answers

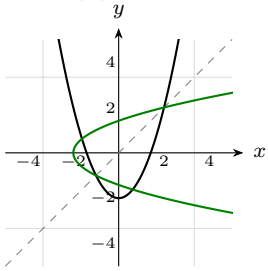
Practice Problems

1. The graph of f is black, the graph of the inverse relation is green, and the dashed line is $y = x$.

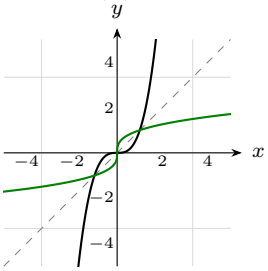
a) $f(x) = \frac{1}{2}x + 1$



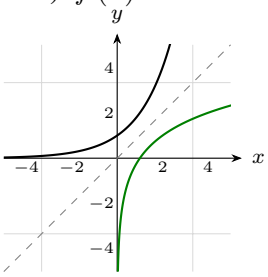
b) $f(x) = x^2 - 2$



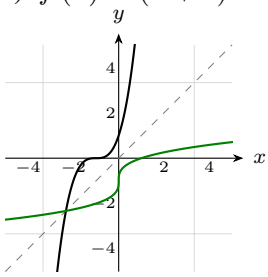
c) $f(x) = x^3$



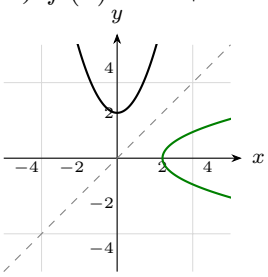
d) $f(x) = 2^x$



e) $f(x) = (x+1)^3$



f) $f(x) = x^2 + 2$



In b) and f), the inverse relation is not a function, because f is not one-to-one.

2. a) $f^{-1}(x) = \dfrac{\log_3 x + 1}{5}$

b) $f^{-1}(x) = \dfrac{5x + 7}{2 - 3x}$

c) $f^{-1}(x) = \dfrac{e^x + 1}{2}$

d) $f^{-1}(x) = \dfrac{1 - x}{3x}$

e) $f^{-1}(x) = \left(\dfrac{2x + 5}{3}\right)^2$

f) $f^{-1}(x) = 2 \cdot 3^x + 14$

g) $f^{-1}(x) = \dfrac{2x + 5}{3x - 2}$

h) $f^{-1}(x) = \log_5\left(\dfrac{x + 3}{2}\right)$

i) $f^{-1}(x) = \dfrac{2x + 1}{4 - x}$

3. a) 7 b) -1 c) 5 d) $\frac{1}{3}$ e) 4 f) -3

4. a) yes b) no; for example, $f(g(8)) = 26 \neq 8$ c) yes

5. a) $f^{-1}(x) = \sqrt{x - 3}$; f: domain $[0, \infty)$, range $[3, \infty)$; f^{-1}: domain $[3, \infty)$, range $[0, \infty)$

b) $f^{-1}(x) = \sqrt{x} + 2$; f: domain $[2, \infty)$, range $[0, \infty)$; f^{-1}: domain $[0, \infty)$, range $[2, \infty)$

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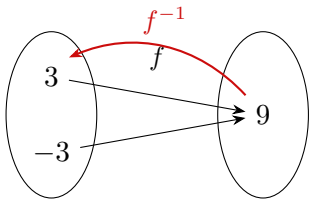
Enrichment



Let $f(x) = x^2$. We restrict its domain to $[0, \infty)$, and then the inverse function is $f^{-1}(x) = \sqrt{x}$.

$f\left(f^{-1}(x)\right) = \left(\sqrt{x}\right)^2 = x$ is still true, but $f^{-1}(f(x)) = \sqrt{x^2} = x$ is false.

Consider $f^{-1}(f(-3)) = \sqrt{(-3)^2} = \sqrt{9} = 3$. We started with -3 and landed on 3. Therefore, this composition of functions is not the identity function. In fact, $f^{-1}(f(x)) = \sqrt{x^2} = |x|$.

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