

Sample Problems

1. Compute each of the following limits.

a) $\lim_{x \rightarrow \infty} 3x^4$ c) $\lim_{x \rightarrow \infty} (-2x^5)$ e) $\lim_{x \rightarrow \infty} \left(-\frac{2}{3}x^6\right)$ g) $\lim_{x \rightarrow \infty} 4x^3$

b) $\lim_{x \rightarrow -\infty} 3x^4$ d) $\lim_{x \rightarrow -\infty} (-2x^5)$ f) $\lim_{x \rightarrow -\infty} \left(-\frac{2}{3}x^6\right)$ h) $\lim_{x \rightarrow -\infty} 4x^3$

2. Compute each of the following limits.

a) $\lim_{x \rightarrow \infty} 2^x$ c) $\lim_{x \rightarrow \infty} \left(\frac{2}{3}\right)^x$ e) $\lim_{x \rightarrow \infty} \frac{2^{x+3}}{3^{x+1}}$ g) $\lim_{x \rightarrow \infty} \frac{2^{2x+1}}{3^{x-1}}$

b) $\lim_{x \rightarrow -\infty} 2^x$ d) $\lim_{x \rightarrow -\infty} \left(\frac{2}{3}\right)^x$ f) $\lim_{x \rightarrow -\infty} \frac{2^{x+3}}{3^{x+1}}$ h) $\lim_{x \rightarrow -\infty} \frac{2^{2x+1}}{3^{x-1}}$

3. Compute each of the following limits.

a) $\lim_{x \rightarrow \infty} \frac{1}{x}$ d) $\lim_{x \rightarrow -\infty} \left(\frac{-5}{2x^3} - 7 + \frac{8}{x}\right)$ g) $\lim_{x \rightarrow \infty} \frac{-5x^3 - 2x + 4}{x^2}$

b) $\lim_{x \rightarrow -\infty} \frac{1}{x}$ e) $\lim_{x \rightarrow \infty} \left(-2x^3 + 1 - \frac{5}{x} + \frac{12}{x^4}\right)$ h) $\lim_{x \rightarrow \infty} \frac{-5x^3 - 2x + 4}{x^3}$

c) $\lim_{x \rightarrow \infty} \frac{-5}{2x^3}$ f) $\lim_{x \rightarrow -\infty} \frac{3x - 2}{x}$ i) $\lim_{x \rightarrow \infty} \frac{-5x^3 - 2x + 4}{x^4}$

4. Compute each of the following limits.

a) $\lim_{x \rightarrow -\infty} (-2x^5 - 8x^4 + 7x^3 - 10)$ c) $\lim_{x \rightarrow -\infty} (-2x^5 + 8x^6)$

b) $\lim_{x \rightarrow \infty} (-2x^5 - 8x^4 + 7x^3 - 10)$ d) $\lim_{x \rightarrow \infty} (-2x^5 + 8x^6)$

5. Compute each of the following limits.

a) $\lim_{x \rightarrow -\infty} \frac{x + x^2 - 6}{6x + 5x^2 + 2x^3}$ b) $\lim_{x \rightarrow \infty} \frac{x^2 + 9}{5x + 2x^2 - 3}$ c) $\lim_{x \rightarrow -\infty} \frac{x^3 - 9x + 1}{3x^2 - 2x - 15}$

Practice Problems

1. Compute each of the following limits.

a) $\lim_{x \rightarrow \infty} \left(-\frac{3}{8}x^{15}\right)$ c) $\lim_{x \rightarrow \infty} \frac{1}{3}x^8$ e) $\lim_{x \rightarrow \infty} 4x^9$ g) $\lim_{x \rightarrow \infty} (-7x^{10})$

b) $\lim_{x \rightarrow -\infty} \left(-\frac{3}{8}x^{15}\right)$ d) $\lim_{x \rightarrow -\infty} \frac{1}{3}x^8$ f) $\lim_{x \rightarrow -\infty} 4x^9$ h) $\lim_{x \rightarrow -\infty} (-7x^{10})$

2. Compute each of the following limits.

a) $\lim_{x \rightarrow \infty} \frac{2^{3x-1}}{5^{x-1}}$

c) $\lim_{x \rightarrow \infty} \frac{2^{2x+3}}{5^{x-1}}$

e) $\lim_{x \rightarrow \infty} \frac{2^{2x+3}}{4^{x-1}}$

g) $\lim_{x \rightarrow \infty} \frac{2^{x+3} \cdot 3^{x-1}}{7^{x-2}}$

b) $\lim_{x \rightarrow -\infty} \frac{2^{3x-1}}{5^{x-1}}$

d) $\lim_{x \rightarrow -\infty} \frac{2^{2x+3}}{5^{x-1}}$

f) $\lim_{x \rightarrow -\infty} \frac{2^{2x+3}}{4^{x-1}}$

h) $\lim_{x \rightarrow -\infty} \frac{2^{2x+3} \cdot 3^{x-1}}{7^{x-2}}$

3. Compute each of the following limits.

a) $\lim_{x \rightarrow \infty} \frac{3}{x^5}$

g) $\lim_{x \rightarrow \infty} \left(5x - \frac{2}{x+3} \right)$

m) $\lim_{x \rightarrow \infty} \frac{-3x^5 + 2x - 5}{x^2}$

b) $\lim_{x \rightarrow -\infty} \frac{3}{x^5}$

h) $\lim_{x \rightarrow -\infty} \left(5x - \frac{2}{x+3} \right)$

n) $\lim_{x \rightarrow -\infty} \frac{-3x^5 + 2x - 5}{x^2}$

c) $\lim_{x \rightarrow \infty} \left(1 - \frac{2}{x} + \frac{5}{3x^4} \right)$

i) $\lim_{x \rightarrow \infty} \frac{5x-3}{x}$

o) $\lim_{x \rightarrow \infty} \frac{-4x^8 + x^3 - x + 7}{x^4}$

d) $\lim_{x \rightarrow -\infty} \left(1 - \frac{2}{x} + \frac{5}{3x^4} \right)$

j) $\lim_{x \rightarrow -\infty} \frac{5x-3}{x}$

p) $\lim_{x \rightarrow -\infty} \frac{-4x^8 + x^3 - x + 7}{x^4}$

e) $\lim_{x \rightarrow \infty} \left(3 + \frac{5}{x^3} - \frac{7}{6x} \right)$

k) $\lim_{x \rightarrow \infty} \frac{1-3x}{2x}$

f) $\lim_{x \rightarrow -\infty} \left(3 + \frac{5}{x^3} - \frac{7}{6x} \right)$

l) $\lim_{x \rightarrow -\infty} \frac{1-3x}{2x}$

4. Compute each of the following limits.

a) $\lim_{x \rightarrow -\infty} (-7x^5 + x^3)$

c) $\lim_{x \rightarrow -\infty} \left(120x^5 - \frac{1}{4}x^6 \right)$

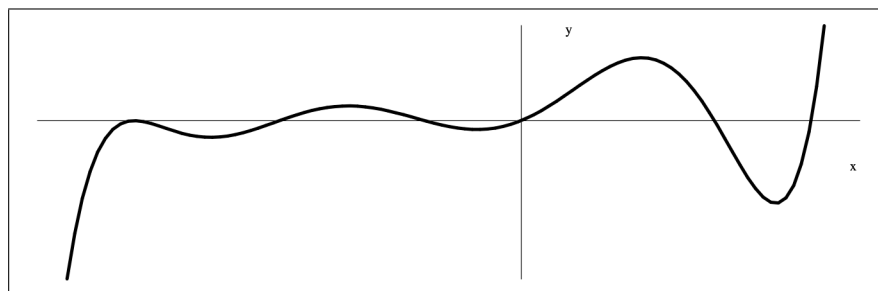
e) $\lim_{x \rightarrow -\infty} \left(8x^4 - 3x^3 - \frac{1}{5}x + 2 \right)$

b) $\lim_{x \rightarrow \infty} (-7x^5 + x^3)$

d) $\lim_{x \rightarrow \infty} \left(120x^5 - \frac{1}{4}x^6 \right)$

f) $\lim_{x \rightarrow \infty} \left(8x^4 - 3x^3 - \frac{1}{5}x + 2 \right)$

5. The graph of a polynomial function is shown on the picture below. What can we state about this polynomial based on its end-behavior?



6. Compute each of the following limits.

a) $\lim_{x \rightarrow -\infty} \frac{1}{x}$

c) $\lim_{x \rightarrow -\infty} \left(2 - \frac{5}{x^3} \right)$

e) $\lim_{x \rightarrow \infty} \frac{2x^2 + 3x + 1}{3x^2 - 5x + 2}$

g) $\lim_{x \rightarrow -\infty} \frac{3x^2 - 1}{5x^2 - 3x + 2}$

b) $\lim_{x \rightarrow -\infty} \frac{-5}{2x^3}$

d) $\lim_{x \rightarrow \infty} \left(3 - \frac{2}{x} + \frac{11}{x^4} \right)$

f) $\lim_{x \rightarrow -\infty} \frac{-3x^3 + 2x + 1}{5x - 3}$

h) $\lim_{x \rightarrow -\infty} \frac{20x - 2x^2 - 42}{5x^3 - 20x^2 - 105x}$

Sample Problems - Answers

1. a) ∞ b) ∞ c) $-\infty$ d) ∞ e) $-\infty$ f) $-\infty$ g) ∞ h) $-\infty$
2. a) ∞ b) 0 c) 0 d) ∞ e) 0 f) ∞ g) ∞ h) 0
3. a) 0 b) 0 c) 0 d) -7 e) $-\infty$ f) 3 g) $-\infty$ h) -5 i) 0
4. a) ∞ b) $-\infty$ c) ∞ d) ∞
5. a) 0 b) $\frac{1}{2}$ c) $-\infty$

Practice Problems - Answers

1. a) $-\infty$ b) ∞ c) ∞ d) ∞ e) ∞ f) $-\infty$ g) $-\infty$ h) $-\infty$
2. a) ∞ b) 0 c) 0 d) ∞ e) 32 f) 32 g) 0 h) ∞
3. a) 0 b) 0 c) 1 d) 1 e) 3 f) 3 g) ∞ h) $-\infty$ i) 5 j) 5 k) $-\frac{3}{2}$ l) $-\frac{3}{2}$
m) $-\infty$ n) ∞ o) $-\infty$ p) $-\infty$
4. a) ∞ b) $-\infty$ c) $-\infty$ d) $-\infty$ e) ∞ f) ∞
5. Since $\lim_{x \rightarrow -\infty} f(x) = -\infty$ and $\lim_{x \rightarrow \infty} f(x) = \infty$, the polynomial is of odd degree and has a positive leading coefficient.
6. a) 0 b) 0 c) 2 d) 3 e) $\frac{2}{3}$ f) $-\infty$ g) $\frac{3}{5}$ h) 0

Sample Problems - Solutions

1. Compute each of the following limits.

a) $\lim_{x \rightarrow \infty} 3x^4$

Solution: Since the limit we are asked for is as x approaches infinity, we should think of x as a very large positive number. Then $3x^4$ is very large, and also positive because it is the product of five positive numbers.

$$3x^4 = \underset{\text{positive}}{3} \cdot \underset{\text{positive}}{x} \cdot \underset{\text{positive}}{x} \cdot \underset{\text{positive}}{x} \cdot \underset{\text{positive}}{x} \cdot \underset{\text{positive}}{x}$$

So the answer is ∞ . We state the answer: $\lim_{x \rightarrow \infty} 3x^4 = \infty$.

b) $\lim_{x \rightarrow -\infty} 3x^4$

Solution: Since the limit we are asked for is as x approaches negative infinity, we should think of x as a very large negative number. Then $3x^4$ is very large, and also positive because it is the product of one positive and four negative numbers.

$$3x^4 = \underset{\text{positive}}{3} \cdot \underset{\text{negative}}{x} \cdot \underset{\text{negative}}{x} \cdot \underset{\text{negative}}{x} \cdot \underset{\text{negative}}{x} \cdot \underset{\text{negative}}{x}$$

So the answer is ∞ . We state the answer: $\lim_{x \rightarrow -\infty} 3x^4 = \infty$

c) $\lim_{x \rightarrow \infty} (-2x^5)$

Solution: Since the limit we are asked for is as x approaches infinity, we should think of x as a very large positive number. Then $-2x^5$ is very large, and also negative because it is the product of one negative and five positive numbers.

$$-2x^5 = \underset{\text{negative}}{-2} \cdot \underset{\text{positive}}{x} \cdot \underset{\text{positive}}{x} \cdot \underset{\text{positive}}{x} \cdot \underset{\text{positive}}{x} \cdot \underset{\text{positive}}{x} \cdot \underset{\text{positive}}{x}$$

So the answer is $-\infty$. We state the answer: $\lim_{x \rightarrow \infty} (-2x^5) = -\infty$

d) $\lim_{x \rightarrow -\infty} (-2x^5)$

Solution: Since the limit we are asked for is as x approaches negative infinity, we should think of x as a very large negative number. Then $-2x^5$ is very large, and also positive because it is the product of six negative numbers.

$$-2x^5 = \underset{\text{negative}}{-2} \cdot \underset{\text{negative}}{x} \cdot \underset{\text{negative}}{x} \cdot \underset{\text{negative}}{x} \cdot \underset{\text{negative}}{x} \cdot \underset{\text{negative}}{x} \cdot \underset{\text{negative}}{x}$$

So the answer is ∞ . We state the answer: $\lim_{x \rightarrow -\infty} (-2x^5) = \infty$

e) $\lim_{x \rightarrow \infty} \left(-\frac{2}{3}x^6\right)$

Solution: Since the limit we are asked for is as x approaches infinity, we should think of x as a very large positive number. Then $-\frac{2}{3}x^6$ is very large, and also negative because it is the product of one negative and six positive numbers.

$$-\frac{2}{3}x^6 = \underset{\text{negative}}{-\frac{2}{3}} \cdot \underset{\text{positive}}{x} \cdot \underset{\text{positive}}{x} \cdot \underset{\text{positive}}{x} \cdot \underset{\text{positive}}{x} \cdot \underset{\text{positive}}{x} \cdot \underset{\text{positive}}{x} \cdot \underset{\text{positive}}{x}$$

So the answer is $-\infty$. We state the answer: $\lim_{x \rightarrow \infty} \left(-\frac{2}{3}x^6\right) = -\infty$

f) $\lim_{x \rightarrow -\infty} \left(-\frac{2}{3}x^6\right)$

Solution: Since the limit we are asked for is as x approaches negative infinity, we should think of x as a very large negative number. Then $-\frac{2}{3}x^6$ is very large, and also negative because it is the product of seven negative numbers.

$$-\frac{2}{3}x^6 = \underset{\text{negative}}{-\frac{2}{3}} \cdot \underset{\text{negative}}{x} \cdot \underset{\text{negative}}{x} \cdot \underset{\text{negative}}{x} \cdot \underset{\text{negative}}{x} \cdot \underset{\text{negative}}{x} \cdot \underset{\text{negative}}{x} \cdot \underset{\text{negative}}{x}$$

So the answer is $-\infty$. We state the answer: $\lim_{x \rightarrow -\infty} -\frac{2}{3}x^6 = -\infty$

g) $\lim_{x \rightarrow \infty} 4x^3$

Solution: Since the limit we are asked for is as x approaches infinity, we should think of x as a very large positive number. Then $4x^3$ is very large, and also positive because it is the product of four positive numbers.

$$4x^3 = \underset{\text{positive}}{4} \cdot \underset{\text{positive}}{x} \cdot \underset{\text{positive}}{x} \cdot \underset{\text{positive}}{x}$$

So the answer is ∞ . We state the answer: $\lim_{x \rightarrow \infty} 4x^3 = \infty$

h) $\lim_{x \rightarrow -\infty} 4x^3$

Solution: Since the limit we are asked for is as x approaches negative infinity, we should think of x as a very large negative number. Then $4x^3$ is very large, and also negative because it is the product of one positive and three negative numbers.

$$4x^3 = \underset{\text{positive}}{4} \cdot \underset{\text{negative}}{x} \cdot \underset{\text{negative}}{x} \cdot \underset{\text{negative}}{x}$$

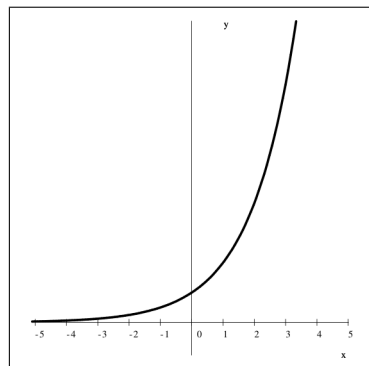
So the answer is $-\infty$. We state the answer: $\lim_{x \rightarrow -\infty} 4x^3 = -\infty$

2. Compute each of the following limits.

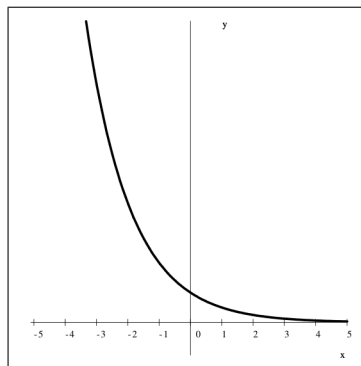
Let $a > 0$. Then the limit of the exponential function $f(x) = a^x$ is as follows.

Case 1. If $a > 1$, then $\lim_{x \rightarrow \infty} a^x = \infty$ and $\lim_{x \rightarrow -\infty} a^x = 0$

Case 2. If $0 < a < 1$, then $\lim_{x \rightarrow \infty} a^x = 0$ and $\lim_{x \rightarrow -\infty} a^x = \infty$



$a > 1$



$0 < a < 1$

a) $\lim_{x \rightarrow \infty} 2^x$ and b) $\lim_{x \rightarrow -\infty} 2^x$

Solution: Since $2 > 1$, these limits are ∞ and 0 , i.e. $\lim_{x \rightarrow \infty} 2^x = \infty$ and $\lim_{x \rightarrow -\infty} 2^x = 0$.

c) $\lim_{x \rightarrow \infty} \left(\frac{2}{3}\right)^x$ and d) $\lim_{x \rightarrow -\infty} \left(\frac{2}{3}\right)^x$

Solution: Since $\frac{2}{3} < 1$, these limits are 0 and ∞ , i.e. $\lim_{x \rightarrow \infty} \left(\frac{2}{3}\right)^x = 0$ and $\lim_{x \rightarrow -\infty} \left(\frac{2}{3}\right)^x = \infty$.

e) $\lim_{x \rightarrow \infty} \frac{2^{x+3}}{3^{x+1}}$

Solution: We start by re-writing the exponential expressions. The goal is to bring it into a form where there is only one exponential expression involving x .

$$\frac{2^{x+3}}{3^{x+1}} = \frac{2^x \cdot 2^3}{3^x \cdot 3^1} = \frac{2^x \cdot 8}{3^x \cdot 3} = \frac{8}{3} \left(\frac{2}{3}\right)^x$$

$$\text{Thus } \lim_{x \rightarrow \infty} \frac{2^{x+3}}{3^{x+1}} = \lim_{x \rightarrow \infty} \frac{8}{3} \left(\frac{2}{3}\right)^x = \frac{8}{3} \lim_{x \rightarrow \infty} \left(\frac{2}{3}\right)^x = 0 \quad \text{since } \frac{2}{3} < 1$$

f) $\lim_{x \rightarrow -\infty} \frac{2^{x+3}}{3^{x+1}}$

$$\text{Solution: } \lim_{x \rightarrow -\infty} \frac{2^{x+3}}{3^{x+1}} = \lim_{x \rightarrow -\infty} \frac{2^{x+3}}{3^{x+1}} = \lim_{x \rightarrow -\infty} \frac{8}{3} \left(\frac{2}{3}\right)^x = \frac{8}{3} \lim_{x \rightarrow -\infty} \left(\frac{2}{3}\right)^x = \infty$$

g) $\lim_{x \rightarrow \infty} \frac{2^{2x+1}}{3^{x-1}}$

Solution: We start by re-writing the exponential expressions. The goal is to bring it into a form where there is only one exponential expression involving x .

$$\frac{2^{2x+1}}{3^{x-1}} = \frac{2^{2x} \cdot 2^1}{\frac{3^x}{3^1}} = \frac{(2^2)^x \cdot 2}{3^x \cdot \frac{1}{3}} = \frac{4^x \cdot 6}{3^x} = 6 \left(\frac{4}{3}\right)^x$$

$$\text{Thus } \lim_{x \rightarrow \infty} \frac{2^{2x+1}}{3^{x-1}} = \lim_{x \rightarrow \infty} 6 \left(\frac{4}{3}\right)^x = 6 \lim_{x \rightarrow \infty} \left(\frac{4}{3}\right)^x = \infty \quad \text{since } \frac{4}{3} > 1$$

h) $\lim_{x \rightarrow -\infty} \frac{2^{2x+1}}{3^{x-1}}$

$$\text{Solution: } \lim_{x \rightarrow -\infty} \frac{2^{2x+1}}{3^{x-1}} = \lim_{x \rightarrow -\infty} 6 \left(\frac{4}{3}\right)^x = 6 \lim_{x \rightarrow -\infty} \left(\frac{4}{3}\right)^x = 0$$

3. Compute each of the following limits.

a) $\lim_{x \rightarrow \infty} \frac{1}{x}$

Solution: This is a very important limit. Since the limit we are asked for is as x approaches infinity, we should think of x as a very large positive number. The reciprocal of a very large positive number is a very small positive number. This limit is 0 .

b) $\lim_{x \rightarrow -\infty} \frac{1}{x}$

Solution: Since the limit we are asked for is as x approaches negative infinity, we should think of x as a very large negative number. The reciprocal of a very large negative number is a very small negative number. This limit is 0 .

c) $\lim_{x \rightarrow \infty} \frac{-5}{2x^3}$

Solution: Since the limit we are asked for is as x approaches infinity, we should think of x as a very large positive number. We divide -5 by a very large positive number. This limit is 0.

d) $\lim_{x \rightarrow -\infty} \left(\frac{-5}{2x^3} - 7 + \frac{8}{x} \right)$

Solution: This limit is -7 since the other two terms approach zero as x approaches negative infinity. Using mathematical notation,

$$\lim_{x \rightarrow -\infty} \frac{-5}{2x^3} - 7 + \frac{8}{x} = \lim_{x \rightarrow -\infty} \frac{-5}{2x^3} + \lim_{x \rightarrow -\infty} -7 + \lim_{x \rightarrow -\infty} \frac{8}{x} = 0 - 7 + 0 = -7$$

e) $\lim_{x \rightarrow \infty} \left(-2x^3 + 1 - \frac{5}{x} + \frac{12}{x^4} \right)$

Solution: This limit is $-\infty$ since the first term approaches negative infinity, the second term approaches 1 and the other two terms approach zero as x approaches infinity. Using mathematical notation,

$$\lim_{x \rightarrow \infty} \left(-2x^3 + 1 - \frac{5}{x} + \frac{12}{x^4} \right) = \lim_{x \rightarrow \infty} (-2x^3) + \lim_{x \rightarrow \infty} 1 + \lim_{x \rightarrow \infty} \left(-\frac{5}{x} \right) + \lim_{x \rightarrow \infty} \left(\frac{12}{x^4} \right) = -\infty + 1 + 0 + 0 = -\infty$$

f) $\lim_{x \rightarrow -\infty} \frac{3x - 2}{x}$

Solution: This problem is similar to the previous problems after a bit of algebra. We simply divide by x and then the limit becomes familiar.

$$\lim_{x \rightarrow -\infty} \frac{3x - 2}{x} = \lim_{x \rightarrow -\infty} \left(\frac{3x}{x} - \frac{2}{x} \right) = \lim_{x \rightarrow -\infty} \left(3 - \frac{2}{x} \right) = 3$$

g) $\lim_{x \rightarrow \infty} \frac{-5x^3 - 2x + 4}{x^2}$

Solution:

$$\begin{aligned} \lim_{x \rightarrow \infty} \frac{-5x^3 - 2x + 4}{x^2} &= \lim_{x \rightarrow \infty} \left(\frac{-5x^3}{x^2} + \frac{-2x}{x^2} + \frac{4}{x^2} \right) = \lim_{x \rightarrow \infty} \left(-5x - \frac{2}{x} + \frac{4}{x^2} \right) \\ &= \lim_{x \rightarrow \infty} (-5x) + \lim_{x \rightarrow \infty} \left(-\frac{2}{x} \right) + \lim_{x \rightarrow \infty} \left(\frac{4}{x^2} \right) = -\infty + 0 + 0 = -\infty \end{aligned}$$

h) $\lim_{x \rightarrow \infty} \frac{-5x^3 - 2x + 4}{x^3}$

Solution:

$$\lim_{x \rightarrow \infty} \frac{-5x^3 - 2x + 4}{x^3} = \lim_{x \rightarrow \infty} \left(\frac{-5x^3}{x^3} + \frac{-2x}{x^3} + \frac{4}{x^3} \right) = \lim_{x \rightarrow \infty} \left(-5 - \frac{2}{x^2} + \frac{4}{x^3} \right) = -5$$

i) $\lim_{x \rightarrow \infty} \frac{-5x^3 - 2x + 4}{x^4}$

Solution:

$$\lim_{x \rightarrow \infty} \frac{-5x^3 - 2x + 4}{x^4} = \lim_{x \rightarrow \infty} \left(\frac{-5x^3}{x^4} + \frac{-2x}{x^4} + \frac{4}{x^4} \right) = \lim_{x \rightarrow \infty} \left(-\frac{5}{x} - \frac{2}{x^3} + \frac{4}{x^4} \right) = 0$$

4. Compute each of the following limits.

a) $\lim_{x \rightarrow -\infty} (-2x^5 - 8x^4 + 7x^3 - 10)$

Solution: The first term, $-2x^5$ approaches infinity and the second term, $-8x^4$ approaches negative infinity. This does not give us enough information about the entire polynomial. A limit like this is called an **indeterminate**. We will bring this expression to a form that is not an indeterminate. In this case, factoring out the first term does the trick.

In case of a polynomial, the limits at infinity and negative infinity are completely determined by its leading term. Recall that the leading term is the highest degree term.

$$\lim_{x \rightarrow -\infty} (-2x^5 - 8x^4 + 7x^3 - 10) = \lim_{x \rightarrow -\infty} (-2x^5)$$

Here is the computation showing why this is true. We first factor out the entire leading term.

$$\begin{aligned} \lim_{x \rightarrow -\infty} (-2x^5 - 8x^4 + 7x^3 - 10) &= \lim_{x \rightarrow -\infty} (-2x^5) \left(1 + \frac{4}{x} - \frac{7}{x^2} + \frac{5}{x^5} \right) \\ &= \lim_{x \rightarrow -\infty} (-2x^5) \cdot \lim_{x \rightarrow -\infty} \left(1 + \frac{4}{x} - \frac{7}{x^2} + \frac{5}{x^5} \right) \\ &= \lim_{x \rightarrow -\infty} (-2x^5) \cdot 1 = \lim_{x \rightarrow -\infty} (-2x^5) \end{aligned}$$

We can now easily determine that this limit is ∞ . (See problem number 1.)

b) $\lim_{x \rightarrow \infty} (-2x^5 - 8x^4 + 7x^3 - 10)$

Solution: **In case of a polynomial, the limits at infinity and negative infinity are completely determined by its leading term.** Recall that the leading term is the highest degree term.

$$\lim_{x \rightarrow \infty} (-2x^5 - 8x^4 + 7x^3 - 10) = \lim_{x \rightarrow \infty} (-2x^5)$$

Here is the computation showing why this is true. We first factor out the entire leading term.

$$\begin{aligned} \lim_{x \rightarrow \infty} (-2x^5 - 8x^4 + 7x^3 - 10) &= \lim_{x \rightarrow \infty} (-2x^5) \left(1 + \frac{4}{x} - \frac{7}{2x^2} + \frac{5}{x^5} \right) \\ &= \lim_{x \rightarrow \infty} (-2x^5) \cdot \lim_{x \rightarrow \infty} \left(1 + \frac{4}{x} - \frac{7}{2x^2} + \frac{5}{x^5} \right) \\ &= \lim_{x \rightarrow \infty} (-2x^5) \cdot 1 = \lim_{x \rightarrow \infty} (-2x^5) \end{aligned}$$

We can now easily determine that this limit is $-\infty$. (See problem number 1.)

c) $\lim_{x \rightarrow -\infty} (-2x^5 + 8x^6)$

Solution: **In case of a polynomial, the limits at infinity and negative infinity are completely determined by its leading term.**

$$\begin{aligned} \lim_{x \rightarrow -\infty} (-2x^5 + 8x^6) &= \lim_{x \rightarrow -\infty} 8x^6 = \infty \quad \text{because} \\ \lim_{x \rightarrow -\infty} (-2x^5 + 8x^6) &= \lim_{x \rightarrow -\infty} (8x^6 - 2x^5) = \lim_{x \rightarrow -\infty} (8x^6) \left(1 - \frac{1}{4x} \right) = \lim_{x \rightarrow -\infty} (8x^6) \cdot \lim_{x \rightarrow -\infty} \left(1 - \frac{1}{4x} \right) \\ &= \lim_{x \rightarrow -\infty} (8x^6) \cdot 1 = \lim_{x \rightarrow -\infty} 8x^6 \end{aligned}$$

We can now easily determine that this limit is ∞ . (See problem number 1.)

d) $\lim_{x \rightarrow \infty} (-2x^5 + 8x^6)$

Solution: **In case of a polynomial, the limits at infinity and negative infinity are completely determined by its leading term.**

$$\begin{aligned}\lim_{x \rightarrow \infty} (-2x^5 + 8x^6) &= \lim_{x \rightarrow \infty} 8x^6 = \infty \quad \text{because} \\ \lim_{x \rightarrow \infty} (-2x^5 + 8x^6) &= \lim_{x \rightarrow \infty} (8x^6 - 2x^5) = \lim_{x \rightarrow \infty} (8x^6) \left(1 - \frac{1}{4x}\right) = \lim_{x \rightarrow \infty} (8x^6) \cdot \lim_{x \rightarrow \infty} \left(1 - \frac{1}{4x}\right) \\ &= \lim_{x \rightarrow \infty} (8x^6) \cdot 1 = \infty\end{aligned}$$

We can now easily determine that this limit is ∞ . (See problem number 1.)

5. Compute each of the following limits.

a) $\lim_{x \rightarrow -\infty} \frac{x + x^2 - 6}{6x + 5x^2 + 2x^3} = 0$

Solution: The numerator approaches infinity and the denominator approaches negative infinity. This does not give us enough information about the quotient. A limit like this is called an **indeterminate**. We will bring this expression to a form that is not an indeterminate. Let us rearrange the polynomials in the rational function given. Then we will factor out the leading term in the numerator and denominator.

$$\lim_{x \rightarrow -\infty} \frac{x^2 + x - 6}{2x^3 + 5x^2 + 6x} = \lim_{x \rightarrow -\infty} \frac{x^2 \left(1 + \frac{1}{x} - \frac{6}{x^2}\right)}{2x^3 \left(1 + \frac{5}{2x} + \frac{3}{x^2}\right)}$$

We now express the limit of the product as the product of two limits

$$\lim_{x \rightarrow -\infty} \frac{x^2 \left(1 + \frac{1}{x} - \frac{6}{x^2}\right)}{2x^3 \left(1 + \frac{5}{2x} + \frac{3}{x^2}\right)} = \lim_{x \rightarrow -\infty} \frac{x^2}{2x^3} \cdot \lim_{x \rightarrow -\infty} \frac{\left(1 + \frac{1}{x} - \frac{6}{x^2}\right)}{\left(1 + \frac{5}{2x} + \frac{3}{x^2}\right)}$$

The first expression can be simplified and thus has a limit we can easily determine its limit. The second expression, although looks unfriendly, is always going to approach 1.

$$\lim_{x \rightarrow -\infty} \frac{x^2}{2x^3} \cdot \lim_{x \rightarrow -\infty} \frac{\left(1 + \frac{1}{x} - \frac{6}{x^2}\right)}{\left(1 + \frac{5}{2x} + \frac{3}{x^2}\right)} = \lim_{x \rightarrow -\infty} \frac{1}{2x} \cdot 1 = 0 \cdot 1 = 0$$

The entire computation should look like this:

$$\begin{aligned}\lim_{x \rightarrow -\infty} \frac{x^2 + x - 6}{2x^3 + 5x^2 + 6x} &= \lim_{x \rightarrow -\infty} \frac{x^2 \left(1 + \frac{1}{x} - \frac{6}{x^2}\right)}{2x^3 \left(1 + \frac{5}{2x} + \frac{3}{x^2}\right)} = \lim_{x \rightarrow -\infty} \frac{x^2}{2x^3} \cdot \lim_{x \rightarrow -\infty} \left(\frac{1 + \frac{1}{x} - \frac{6}{x^2}}{1 + \frac{5}{2x} + \frac{3}{x^2}} \right) \\ &= \lim_{x \rightarrow -\infty} \frac{1}{2x} \cdot 1 = 0 \cdot 1 = 0\end{aligned}$$

$$\text{b) } \lim_{x \rightarrow \infty} \frac{x^2 + 9}{5x + 2x^2 - 3} = \frac{1}{2}$$

Solution: Both numerator and denominator approach infinity. This does not give us enough information about the quotient. A limit like this is called an **indeterminate**. We will bring this expression to a form that is not an indeterminate. Let us rearrange the polynomials in the rational function given. Then we will factor out the leading term in the numerator and denominator.

$$\begin{aligned} \lim_{x \rightarrow \infty} \frac{x^2 + 9}{2x^2 + 5x - 3} &= \lim_{x \rightarrow \infty} \frac{x^2 \left(1 + \frac{9}{x^2}\right)}{2x^2 \left(1 + \frac{5}{2x} - \frac{3}{2x^2}\right)} = \lim_{x \rightarrow \infty} \frac{x^2}{2x^2} \cdot \lim_{x \rightarrow \infty} \frac{1 + \frac{9}{x^2}}{1 + \frac{5}{2x} - \frac{3}{2x^2}} \\ &= \lim_{x \rightarrow \infty} \frac{1}{2} \cdot \lim_{x \rightarrow \infty} \frac{1 + \frac{9}{x^2}}{1 + \frac{5}{2x} - \frac{3}{2x^2}} = \frac{1}{2} \cdot 1 = \frac{1}{2} \end{aligned}$$

$$\text{c) } \lim_{x \rightarrow -\infty} \frac{x^3 - 9x + 1}{3x^2 - 2x - 15} = -\infty$$

Solution:

$$\begin{aligned} \lim_{x \rightarrow -\infty} \frac{x^3 - 9x + 1}{3x^2 - 2x - 15} &= \lim_{x \rightarrow -\infty} \frac{x^3 \left(1 - \frac{9}{x^2} + \frac{1}{x^3}\right)}{3x^2 \left(1 - \frac{2}{3x} - \frac{5}{x^2}\right)} = \lim_{x \rightarrow -\infty} \frac{x^3}{3x^2} \cdot \lim_{x \rightarrow -\infty} \frac{1 - \frac{9}{x^2} + \frac{1}{x^3}}{\left(1 - \frac{2}{3x} - \frac{5}{x^2}\right)} \\ &= \left(\lim_{x \rightarrow -\infty} \frac{x^3}{3x^2}\right) \cdot 1 = \left(\lim_{x \rightarrow -\infty} \frac{x}{3}\right) \cdot 1 = -\infty \cdot 1 = -\infty \end{aligned}$$