

Definition: **The symbol $\log_a b$ represents the number that if we write as an exponent of a , we achieve b .** This expression is only meaningful if both a and b are positive numbers and $a \neq 1$.

Every logarithmic statement can be re-written as an exponential statement.

$$\log_a b = x \text{ is the same as } a^x = b$$

Example: Find the value of $\log_2 8$.

$\log_2 8$ is a number, it is the exponent we need to write over 2 to obtain 8. In short, $2^? = 8$. The answer is clearly 3, since $2^3 = 8$.

$$\log_2 8 = 3$$

Similarly, most exponential statements can be re-written as a logarithmic statement.

$$n^y = M \text{ is the same as } \log_n M = y$$

Some properties of logarithms follow immediately from its definition.

Theorem 1. $\log_a a^k = k$

Proof. To achieve a^k , we need to raise a exactly to the k th power. Or, using exponential notation,

$$\text{if } x = \log_a a^k, \text{ then } a^x = a^k \implies x = k$$

Theorem 2. $a^{\log_a b} = b$

Proof. Recall the definition of $\log_a b$. This is the number we need to use as an exponent of a to obtain b . In other words,

$$\text{if } x = \log_a b, \text{ then } a^x = b \implies a^{\log_a b} = b$$

Sample Problems

1. Find the exact value of each of the following logarithmic expressions.

a) $\log_3 3$ b) $\log_4 1$ c) $\log_5 \left(\frac{1}{25}\right)$ d) $\log_9 3$ e) $\log_8 \left(\frac{1}{16}\right)$ f) $\ln e^8$

2. Simplify each of the following expressions. Assume that $a, X > 0$ and $a \neq 1$.

a) $\log_a (a^9)$ c) $16^{\log_4 a}$ e) $2^{\log_8 a}$ g) $10^{3 \log_{10} X}$
b) $8^{\log_2 5}$ d) $5^{\log_{25} 3}$ f) $e^{-5 \ln B}$ h) $10^{M \log_{100} 3}$

3. Place each of the following logarithms between two consecutive integers.

a) $\log_3 60$ b) $\log_2 60$ c) $\log_{10} 2011$

4. Solve each of the following equations for x . Present the exact value of each solution.

a) $\log_5 x = 3$ b) $\frac{1}{2} \log_2 (3p + 1) - 1 = 2$ c) $3 \ln (w - 4) - 5 = 1$

5. Prove that if $\log_a b \neq 0$, then $\log_a b$ and $\log_b a$ are reciprocals of each other.

6. What is the connection between $\log_a b$ and $\log_a \left(\frac{1}{b}\right)$?

Practice Problems

1. Find the exact value of each of the following logarithmic expressions.

- | | | | |
|---------------------------------------|-------------------------------------|--------------------------------------|---|
| a) $\log_2 32$ | g) $\log_8 4$ | m) $\log_3 \left(\frac{1}{3}\right)$ | s) $\log_8 \left(\frac{1}{4}\right)$ |
| b) $\log_3 81$ | h) $\log_{10} 0.000001$ | n) $\log_3 0$ | t) $\log_2 8$ |
| c) $\log_4 4$ | i) $\ln \left(\frac{1}{e^2}\right)$ | o) $\log_3 (-3)$ | u) $\log_8 2$ |
| d) $\log_5 1$ | j) $\log_3 9$ | p) $\log_9 3$ | v) $\log_{81} 3$ |
| e) $\log_6 \left(\frac{1}{36}\right)$ | k) $\log_3 3$ | q) $\log_{1/3} 9$ | w) $\log_8 16$ |
| f) $\log_{49} 7$ | l) $\log_3 1$ | r) $\log_7 (7^{23})$ | x) $\log_{27} \left(\frac{1}{9}\right)$ |

2. Simplify each of the following expressions. Assume that $m > 0$ and $m \neq 1$.

- | | | | |
|-------------------|------------------------|--|---------------------------------------|
| a) $3^{\log_3 2}$ | d) $8^{\log_2 7}$ | g) $3^{\log_9 m}$ | j) $8^{\log_{(1/2)} m}$ |
| b) $6^{\log_6 m}$ | e) $0.1^{\log_{10} m}$ | h) $1000^{\log_{10} m}$ | k) $\left(\frac{1}{e}\right)^{\ln 9}$ |
| c) $e^{\ln 12}$ | f) $25^{\log_5(2m)}$ | i) $\left(\frac{1}{2}\right)^{\log_4 m}$ | l) $e^{-\ln(3m)}$ |

3. Place each of the following logarithms between two consecutive integers.

- | | | | |
|----------------|---------------------|------------------------|--------------------------------------|
| a) $\log_2 10$ | c) $\log_3 10$ | e) $\log_{10} 38\,000$ | g) $\log_3 100$ |
| b) $\log_5 10$ | d) $\log_{10} 2011$ | f) $\log_2 100$ | h) $\log_2 \left(\frac{1}{9}\right)$ |

4. Re-write each of the following logarithmic statements as an exponential statement.

- | | | |
|--------------------|--------------------|---------------------|
| a) $\log_3 x = 10$ | c) $\log_5 10 = y$ | e) $\log_A B = C$ |
| b) $\ln A = -2$ | d) $\log_x y = 4$ | f) $\ln(x - 2) = 5$ |

5. Re-write each of the following exponential statements as a logarithmic statement.

- | | | | |
|--------------|--------------|-------------------|---------------|
| a) $A^B = C$ | b) $3^x = 8$ | c) $2^{t-5} = 11$ | d) $e^x = 10$ |
|--------------|--------------|-------------------|---------------|

6. Solve each of the following equations for x . Present the exact value of each solution.

- | | | |
|-------------------------|-----------------------------|--|
| a) $\log_3 x = 2$ | d) $\log_7 (5x - 1) = 2$ | g) $\frac{2}{3} \ln(x - 1) - 1 = 5$ |
| b) $\log_2 (x - 5) = 3$ | e) $\log_3 (2x^2 - 5) = 3$ | h) $\frac{\log_5 (x + 1) - 1}{3} = -1$ |
| c) $\ln(x - 1) = 2$ | f) $\log_{10} (x + 2) = -2$ | |

Sample Problems - Answers

1. a) 1 b) 0 c) -2 d) $\frac{1}{2}$ e) $-\frac{4}{3}$ f) 8
2. a) 9 b) 125 c) a^2 d) $\sqrt{3}$ e) $\sqrt[3]{a}$ f) $\frac{1}{B^5}$ g) X^3 h) $(\sqrt{3})^M = 3^{\frac{M}{2}}$
3. a) $3 < \log_3 60 < 4$ b) $5 < \log_2 60 < 6$ c) $3 < \log_{10} 2011 < 4$
4. a) 125 b) 21 c) $e^2 + 4$
5. see solutions
6. see solutions

Practice Problems - Answers

1. a) 5 b) 4 c) 1 d) 0 e) -2 f) $\frac{1}{2}$ g) $\frac{2}{3}$ h) -6 i) -2 j) 2
- k) 1 l) 0 m) -1 n) undefined o) undefined p) $\frac{1}{2}$ q) -2 r) 23
- s) $-\frac{2}{3}$ t) 3 u) $\frac{1}{3}$ v) $\frac{1}{4}$ w) $\frac{4}{3}$ x) $-\frac{2}{3}$
2. a) 2 b) m c) 12 d) 343 e) $\frac{1}{m}$ f) $4m^2$ g) \sqrt{m} h) m^3 i) $\frac{1}{\sqrt{m}}$ j) $\frac{1}{m^3}$
- k) $\frac{1}{9}$ l) $\frac{1}{3m}$
3. a) $3 < \log_2 10 < 4$ b) $1 < \log_5 10 < 2$ c) $2 < \log_3 10 < 3$ d) $3 < \log 2010 < 4$
- e) $4 < \log_{10} 38\,000 < 5$ f) $6 < \log_2 100 < 7$ g) $4 < \log_3 100 < 5$ h) $-3 < \log_2 \left(\frac{1}{6}\right) < -2$
4. a) $3^{10} = x$ b) $e^{-2} = A$ c) $5^y = 10$ d) $x^4 = y$ e) $A^C = B$ f) $e^5 = x - 2$
5. a) $\log_A C = B$ b) $x = \log_3 8$ c) $t - 5 = \log_2 11$ d) $x = \ln 10$
6. a) 9 b) 13 c) $e^2 + 1$ d) 10 e) 4, -4 f) -1.99 g) $e^9 + 1$ h) $-\frac{24}{25}$

Sample Problems - Solutions

1. Find the exact value of each of the following logarithmic expressions.

a) $\log_3 3$

Solution: $\log_3 3$ is a number, it is the exponent we need to write over 3 to obtain 3.

In short, $3^? = 3$. The answer is clearly 1, since $3^1 = 3$.

$$\log_3 3 = 1$$

b) $\log_4 1$

Solution: $\log_4 1$ is a number, it is the exponent we need to write over 4 to obtain 1.

In short, $4^? = 1$. The answer is clearly 0, since $4^0 = 1$.

$$\log_4 1 = 0$$

c) $\log_5 \left(\frac{1}{25} \right)$

Solution: $\log_5 \left(\frac{1}{25} \right)$ is a number, it is the exponent we need to write over 5 to obtain $\frac{1}{25}$.

In short, $5^? = \frac{1}{25}$. The answer is clearly -2 , since $5^{-2} = \frac{1}{25}$.

$$\log_5 \left(\frac{1}{25} \right) = -2$$

d) $\log_9 3$

Solution: $\log_9 3$ is a number, it is the exponent we need to write over 9 to obtain 3.

In short, $9^? = 3$. We know that 3 is the square root of 9. We re-write this statement as an exponential statement

$$3 = \sqrt{9} = 9^{1/2} \quad \text{and so} \quad \log_9 3 = \frac{1}{2}$$

e) $\log_8 \left(\frac{1}{16} \right)$

Solution: $\log_8 \left(\frac{1}{16} \right)$, it is the exponent we need to write over 8 to obtain $\frac{1}{16}$.

In short, $8^? = \frac{1}{16}$. It looks like we have a problem here, $\frac{1}{16}$ is not a straightforward power of 8. The trick is to realize that they are both powers of 2, and express 2 first as a power of 8.

$$8^{1/3} = 2 \quad \text{and we also know that} \quad \frac{1}{16} = 2^{-4}$$

$$\text{our plan:} \quad 8 \longrightarrow 2 \longrightarrow \frac{1}{16}$$

$$8^{1/3} = 2 \quad \text{raise both sides to the power of } -4$$

$$\left(8^{1/3} \right)^{-4} = 2^{-4} \quad \text{recall that } (a^n)^m = a^{nm}$$

$$8^{-4/3} = \frac{1}{16} \quad \text{and so} \quad \log_8 \left(\frac{1}{16} \right) = -\frac{4}{3}$$

f) $\ln e^8$

Solution: \ln is short for \log_e . So the question is $\log_e (e^8)$. In order to obtain the product e^8 , we will need eighth factors of e . This $\ln e^8 = 8$. (If this problem is confusing because of the unusual number e , try to argue why $\log_3 (3^{25})$ must be 25.)

2. Simplify each of the following expressions. Assume that $a > 0$ and $a \neq 1$.

a) $\log_a (a^9) = 9$

Solution: In order to obtain the product a^9 , we will need nine factors of a .

b) $8^{\log_2 5} = 125$

Solution: In order to simplify this expression, we will use the fact that $2^{\log_2 5} = 5$. Our expression is not exactly the same as $2^{\log_2 5}$ but it is related to it, since $8 = 2^3$. We will use the following rule of exponents:

$$(a^n)^m = a^{nm} = a^{mn} = (a^m)^n$$

$$8^{\log_2 5} = (2^3)^{\log_2 5} = 2^{3 \log_2 5} = 2^{(\log_2 5) \cdot 3} = (2^{\log_2 5})^3 = 5^3 = 125$$

c) $16^{\log_4 a} = a^2$

Solution: In order to simplify this expression, we will use the fact that $4^{\log_4 a} = a$. Our expression is not exactly the same as $4^{\log_4 a}$ but it is related to it, since $16 = 4^2$. We will use the following rule of exponents:

$$(a^n)^m = a^{nm} = a^{mn} = (a^m)^n$$

$$16^{\log_4 a} = (4^2)^{\log_4 a} = 4^{2 \log_4 a} = 4^{(\log_4 a) \cdot 2} = (4^{\log_4 a})^2 = a^2$$

d) $5^{\log_{25} 3} = \sqrt{3}$

Solution: In order to simplify this expression, we will use the fact that $25^{\log_{25} 3} = 3$. Our expression is not exactly the same as $25^{\log_{25} 3}$ but it is related to it, since $5 = 25^{1/2}$. We will use the following rule of exponents:

$$(a^n)^m = a^{nm} = a^{mn} = (a^m)^n$$

$$5^{\log_{25} 3} = (25^{1/2})^{\log_{25} 3} = 25^{(1/2) \log_{25} 3} = 25^{(\log_{25} 3)(1/2)} = (25^{\log_{25} 3})^{1/2} = 3^{1/2} = \sqrt{3}$$

e) $2^{\log_8 a} = \sqrt[3]{a}$

Solution: In order to simplify this expression, we will use the fact that $8^{\log_8 a} = a$. Our expression is not exactly the same as $8^{\log_8 a}$ but it is related to it, since $2 = 8^{1/3}$. We will use the following rule of exponents:

$$(a^n)^m = a^{nm} = a^{mn} = (a^m)^n$$

$$2^{\log_8 a} = (8^{1/3})^{\log_8 a} = 8^{(1/3) \log_8 a} = 8^{(\log_8 a)(1/3)} = (8^{\log_8 a})^{1/3} = a^{1/3} = \sqrt[3]{a}$$

f) $e^{-5 \ln B}$

Solution: In order to simplify this expression, we will use the fact that $e^{\ln B} = B$. Our expression is not exactly the same as $e^{\ln B}$ but it is related to it. We will use the following rule of exponents:

$$(a^n)^m = a^{nm} = a^{mn} = (a^m)^n$$

$$e^{-5 \ln B} = e^{\ln B \cdot (-5)} = (e^{\ln B})^{-5} = B^{-5} = \frac{1}{B^5}$$

g) $10^{3 \log_{10} X}$

Solution: In order to simplify this expression, we will use the fact that $10^{\log_{10} X} = X$. We will use the following rule of exponents:

$$(a^n)^m = a^{nm} = a^{mn} = (a^m)^n$$

$$10^{3 \log_{10} X} = 10^{\log_{10} X \cdot (3)} = \left(10^{\log_{10} X}\right)^3 = X^3$$

h) $10^{M \log_{100} 3}$

Solution: In order to simplify this expression, we will use the fact that $100^{\log_{100} 3} = 3$. We will use the following rule of exponents:

$$(a^n)^m = a^{nm} = a^{mn} = (a^m)^n$$

We also have to re-write 10 as a power of 100. Clearly, $10 = 100^{\frac{1}{2}}$

$$10^{M \log_{100} 3} = \left(100^{1/2}\right)^{M \log_{100} 3} = 100^{\frac{1}{2} M \log_{100} 3} = \left(100^{\log_{100} 3}\right)^{\frac{1}{2} M} = \left(3^{\frac{1}{2}}\right)^M = \left(\sqrt{3}\right)^M = 3^{\frac{M}{2}}$$

3. Place each of the following logarithms between two consecutive integers.

a) $\log_3 60$

Solution: We need to place 60 between two consecutive three-powers. Those are 27 and 81. Since 60 is between 27 and 81, $\log_3 60$ is between 3 and 4.

$$27 < 60 < 81 \implies 3^3 < 60 < 3^4 \implies 3 < \log_3 60 < 4$$

b) $\log_2 60$

Solution: We need to place 60 between two consecutive two-powers. Those are 32 and 64. Since 60 is between 32 and 64, $\log_2 60$ is between 5 and 6.

$$\begin{aligned} 32 &< 60 < 64 \\ 2^5 &< 60 < 2^6 \\ 5 &< \log_2 60 < 6 \end{aligned}$$

c) $\log_{10} 2011$

Solution: We need to place 2011 between two consecutive ten-powers. Those are 1000 and 10 000. Since 2011 is between 1000 and 10 000, $\log_{10} 2011$ is between 3 and 4.

$$\begin{aligned} 1000 &< 2011 < 10\,000 \\ 10^3 &< 2011 < 10^4 \\ 3 &< \log_{10} 2011 < 4 \end{aligned}$$

4. Solve each of the following equations for x . Present the exact value of each solution.

a) $\log_5 x = 3$

Solution: We re-write the logarithmic statement as an exponential statement.

$$\begin{aligned}\log_5 x &= 3 \\ 5^3 &= x \\ 125 &= x\end{aligned}$$

We check: $\log_5 125 = 3$ is indeed true.

b) $\frac{1}{2} \log_2 (3p + 1) - 1 = 2$

Solution: We first isolate the logarithmic expression. Then we re-write the logarithmic statement as an exponential statement, and solve for p .

$$\begin{aligned}\frac{1}{2} \log_2 (3p + 1) - 1 &= 2 && \text{add 1} \\ \frac{1}{2} \log_2 (3p + 1) &= 3 && \text{multiply by 2} \\ \log_2 (3p + 1) &= 6 && \text{re-write into an exponential statement} \\ 3p + 1 &= 2^6 \\ 3p + 1 &= 64 && \text{subtract 1} \\ 3p &= 63 && \text{divide by 3} \\ p &= 21\end{aligned}$$

We check: if $p = 21$, then

$$\text{LHS} = \frac{1}{2} \log_2 (3 \cdot 21 + 1) - 1 = \frac{1}{2} \log_2 (63 + 1) - 1 = \frac{1}{2} (\log_2 64) - 1 = \frac{1}{2} \cdot 6 - 1 = 3 - 1 = 2 = \text{RHS}$$

c) $3 \ln (w - 4) - 5 = 1$

Solution: We first isolate the logarithmic expression. Then we re-write the logarithmic statement as an exponential statement, and solve for w .

$$\begin{aligned}3 \ln (w - 4) - 5 &= 1 && \text{add 5} \\ 3 \ln (w - 4) &= 6 && \text{divide by 3} \\ \ln (w - 4) &= 2 && \text{re-write into an exponential statement} \\ w - 4 &= e^2 && \text{add 4} \\ w &= e^2 + 4\end{aligned}$$

We check: if $w = e^2 + 4$, then

$$\text{LHS} = 3 \ln ((e^2 + 4) - 4) - 5 = 3 \ln (e^2) - 5 = 3 \cdot 2 - 5 = 1 = \text{RHS}$$

5. Claim: If $\log_a b \neq 0$, then $\log_a b$ and $\log_b a$ are reciprocals of each other.

Proof. Let $x = \log_a b$. Then the statement can be re-written as $a^x = b$.

$$\begin{aligned}x &= \log_a b \\ a^x &= b && \text{raise both sides to the power of } \frac{1}{x} \\ (a^x)^{1/x} &= b^{1/x} && (a^x)^{1/x} = a^{x \cdot \frac{1}{x}} = a^1 = a \\ a &= b^{1/x} && \text{same as } \frac{1}{x} = \log_b a \\ \frac{1}{x} &= \log_b a\end{aligned}$$

6. $\log_a b$ and $\log_a \left(\frac{1}{b}\right)$ are opposites of each other.

Proof. Let $x = \log_a b$. Then the statement can be re-written as $a^x = b$.

$$\begin{aligned}x &= \log_a b \\a^x &= b && \text{raise both sides to the power of } -1 \\(a^x)^{-1} &= b^{-1} && (a^x)^{-1} = a^{x \cdot (-1)} = a^{-x} \\a^{-x} &= \frac{1}{b} && \text{same as } -x = \log_a \left(\frac{1}{b}\right) \\-x &= \log_a \left(\frac{1}{b}\right)\end{aligned}$$