This is the second part of studying logarithms. Before we start, let us recall a few things from the first part.

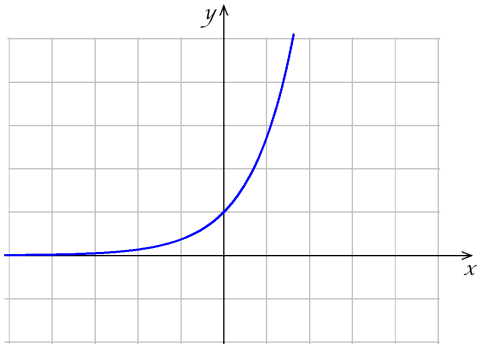
Review: the definition of the logarithm

The symbol $\log_2 8$ represents the number of 2-factors we need to multiply together so that the product is 8. We need three: $2 \cdot 2 \cdot 2 = 8$, so

$$\log_2 8 = 3 \quad \text{because} \quad 2^3 = 8$$

So, in a sense, logarithms are exponents. In general, $\log_a b = x$ because $a^x = b$. Every logarithmic statement can be re-written as an exponential statement, and the number a is called the **base** of the logarithm.

When we extended the concept of exponents from the integers to the real numbers, we ran into trouble with negative bases. So the base of a logarithm must be positive, and it cannot be 1, because every power of 1 is 1. The function $f(x) = 2^x$ has only positive values, so 0 and negative numbers cannot be written as powers of 2. Therefore $\log_a b$ is only defined if b is positive.



$f(x) = 2^x$

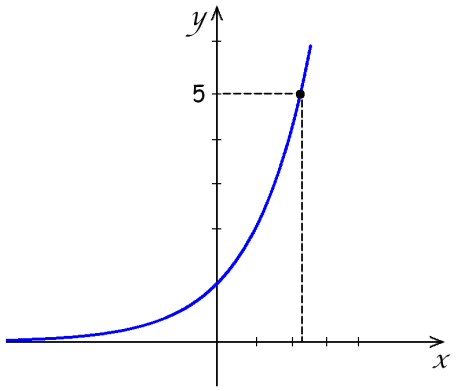
Definition. Suppose that $a, b > 0$ and $a \neq 1$. The symbol $\log_a b$ represents the number of a-factors needed to multiply each other, so that the product is b:

$$\log_a b = x \qquad \text{means exactly the same as} \qquad a^x = b$$

$\log a$ is short for $\log_{10} a$ (the **common logarithm**), and $\ln a$ is short for $\log_e a$ (the **natural logarithm**).

Naturally, we did not need logarithms for a number like $\log_2 8$. We needed them for numbers such as $\log_2 5$. Because $f(x) = 2^x$ is a continuous function, it must take the value 5 somewhere. The x-coordinate of that point is $\log_2 5$. This is an irrational number, and there is no other way to write its exact value.

We saw two theorems as immediate consequences of the definition.



$f(x) = 2^x$

Theorem 1. If $a, b > 0$, $a \neq 1$, and k is any real number, then $\log_a \left(a^k\right) = k$.

Theorem 2. If $a, b > 0$ and $a \neq 1$, then $a^{\log_a b} = b$.

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Example 1. Simplify each of the following expressions. a) $e^{-3\ln 5}$ b) $8^{\log_2 x}$

Solution: Recall that $a^{\log_a b} = b$. Thus $e^{\ln x} = x$ and $2^{\log_2 x} = x$. The trick is to match the base of the exponentiation with the base of the logarithm.

$$\text{a) } e^{-3\ln 5} = \left(e^{\ln 5}\right)^{-3} = 5^{-3} = \frac{1}{125} \qquad \text{b) } 8^{\log_2 x} = \left(2^3\right)^{\log_2 x} = 2^{3\log_2 x} = \left(2^{\log_2 x}\right)^3 = x^3$$

The product rule

The 'ingredients' of the next theorem are the first rule of exponents, $a^n \cdot a^m = a^{n+m}$, and the fact that $a^x = a^y$ implies $x = y$ (because $f(x) = a^x$ is a one-to-one function).

Theorem 3 (the product rule). When both sides exist, $\log_a x + \log_a y = \log_a (xy)$.

Proof. Consider $a^{\log_a x + \log_a y}$ and $a^{\log_a (xy)}$. Since $a^{\log_a b} = b$, $a^{\log_a (xy)} = xy$. The other expression is

$$a^{\log_a x + \log_a y} = a^{\log_a x} \cdot a^{\log_a y} = xy$$

Therefore $a^{\log_a x + \log_a y} = a^{\log_a (xy)}$. Since $f(x) = a^x$ is a one-to-one function, two a-powers can only be equal if their exponents are the same. Therefore $\log_a x + \log_a y = \log_a (xy)$. This completes our proof.

Discussion: Consider the condition 'when both sides exist' in the product rule. When do both sides exist? Is it possible for only one side to exist, but not the other one?

Example 2. Simplify $\log_6 4 + \log_6 54$.

Solution: $\log_6 4 + \log_6 54 = \log_6 (4 \cdot 54) = \log_6 216 = 3$, because $6^3 = 216$.

The quotient rule

This time the ingredient is the second rule of exponents, $\dfrac{a^n}{a^m} = a^{n-m}$.

Theorem 4 (the quotient rule). When both sides exist, $\log_a x - \log_a y = \log_a \left(\dfrac{x}{y}\right)$.

Proof. Consider $a^{\log_a x - \log_a y}$ and $a^{\log_a (x/y)}$. Since $a^{\log_a b} = b$, $a^{\log_a (x/y)} = \dfrac{x}{y}$. The other expression is

$$a^{\log_a x - \log_a y} = \frac{a^{\log_a x}}{a^{\log_a y}} = \frac{x}{y}$$

The two a-powers are equal, so their exponents are the same: $\log_a x - \log_a y = \log_a \left(\dfrac{x}{y}\right)$.

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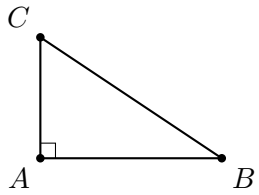
Example 3. a) Simplify $\log_4 320 - \log_4 5$.

b) Suppose that x and y are positive. Find the area of the triangle with vertices $A\left(\log_2 x, \log_3 y\right)$, $B\left(\log_2 (8x), \log_3 y\right)$, and $C\left(\log_2 x, \log_3 (9y)\right)$.

Solution: a) $\log_4 320 - \log_4 5 = \log_4 \dfrac{320}{5} = \log_4 64 = 3$, because $4^3 = 64$.

b) Points A and B have the same second coordinate, so the side AB is horizontal. Points A and C have the same first coordinate, so the side AC is vertical. So the triangle is a right triangle, and the right angle is at A.



We find the lengths of the two legs. Since $8x > x$, the point B is to the right of A, and since $9y > y$, the point C is above A.

$$AB = \log_2 (8x) - \log_2 x = \log_2 \frac{8x}{x} = \log_2 8 = 3 \qquad AC = \log_3 (9y) - \log_3 y = \log_3 \frac{9y}{y} = \log_3 9 = 2$$

The area of the right triangle is $\dfrac{AB \cdot AC}{2} = \dfrac{3 \cdot 2}{2} = 3$. Notice that x and y cancel out: the triangle can be anywhere, but its area is always 3.

The power rule

The ingredient is the third rule of exponents, $(a^n)^m = a^{nm}$.

Theorem 5 (the power rule). When both sides exist, $\log_a (x^y) = y \log_a x$.

Proof. Consider $a^{\log_a (x^y)}$ and $a^{y \log_a x}$. Since $a^{\log_a b} = b$, $a^{\log_a (x^y)} = x^y$. The other expression is

$$a^{y \log_a x} = \left(a^{\log_a x}\right)^y = x^y$$

The two a-powers are equal, so their exponents are the same: $\log_a (x^y) = y \log_a x$. This completes our proof.

Example 4. Simplify each of the following expressions. a) $2 \log_{10} (2x) + \log_{10} (25x)$ b) $1 + 2 \log_2 3 - \log_2 36$

Solution: a) By the power rule, $2 \log_{10} (2x) = \log_{10} \left[(2x)^2\right] = \log_{10} \left(4x^2\right)$. Then we use the product rule.

$$2 \log_{10} (2x) + \log_{10} (25x) = \log_{10} \left(4x^2 \cdot 25x\right) = \log_{10} \left(100x^3\right) = \log_{10} 100 + \log_{10} \left(x^3\right) = 2 + 3 \log_{10} x$$

b) We re-write each term as a base 2 logarithm: $1 = \log_2 2$ and $2 \log_2 3 = \log_2 \left(3^2\right) = \log_2 9$.

$$1 + 2 \log_2 3 - \log_2 36 = \log_2 2 + \log_2 9 - \log_2 36 = \log_2 \frac{2 \cdot 9}{36} = \log_2 \frac{1}{2} = -1$$

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The change-base theorem

A scientific calculator can compute two logarithms: $\log a = \log_{10} a$ and $\ln a = \log_e a$. How could we compute a number such as $\log_2 7$? Let us denote $\log_2 7$ by x. Then

$$\begin{aligned} 2^x &= 7 && \text{take the common logarithm of both sides} \\ \log\left(2^x\right) &= \log 7 && \text{use the power rule} \\ x \log 2 &= \log 7 && \text{divide both sides by } \log 2 \\ x &= \frac{\log 7}{\log 2} \approx 2.8074 \end{aligned}$$

The same computation works with any base c instead of 10. If $\log_a b = x$, then $a^x = b$, so $\log_c\left(a^x\right) = \log_c b$, $x \log_c a = \log_c b$, and $x = \dfrac{\log_c b}{\log_c a}$. This proves our last theorem.

Theorem 6 (the change-base theorem). For all $a, b, c > 0$, $a, c \neq 1$, $\quad \log_a b = \dfrac{\log_c b}{\log_c a}$.

Example 5. Prove each of the following. Assume that $a, b > 0$, $a, b \neq 1$, and $n \neq 0$.

a) $\log_{(a^n)}\left(b^n\right) = \log_a b$ b) $\log_b a = \dfrac{1}{\log_a b}$

Solution: a) We switch to base a:

$$\log_{(a^n)}\left(b^n\right) = \frac{\log_a\left(b^n\right)}{\log_a\left(a^n\right)} = \frac{n \log_a b}{n} = \log_a b$$

In words: if we raise both the base and the number to the same power, the logarithm does not change. For example, $\log_2 5 = \log_4 25$.

b) Again we switch to base a: $\log_b a = \dfrac{\log_a a}{\log_a b} = \dfrac{1}{\log_a b}$. If we swap the two numbers in a logarithm, we get the reciprocal of the original logarithm. This explains something we already know: $\log_2 8 = 3$ and $\log_8 2 = \dfrac{1}{3}$.

Example 6. Simplify each of the following expressions.

a) $\log_9 \sqrt{27}$ b) $3^{\log_9 x}$ c) $\left(\log_3 4\right)\left(\log_4 5\right)\left(\log_5 6\right)\left(\log_6 7\right)\left(\log_7 8\right)\left(\log_8 9\right)$

Solution: a) We switch to base 3: $\log_9 \sqrt{27} = \dfrac{\log_3 \sqrt{27}}{\log_3 9} = \dfrac{\log_3\left(3^{3/2}\right)}{2} = \dfrac{3/2}{2} = \dfrac{3}{4}$

b) We can change the base of the exponentiation, $3^{\log_9 x} = \left(9^{1/2}\right)^{\log_9 x} = 9^{\frac{1}{2}\log_9 x} = \left(9^{\log_9 x}\right)^{1/2} = x^{1/2} = \sqrt{x}$, or the base of the logarithm: $\log_9 x = \dfrac{\log_3 x}{\log_3 9} = \dfrac{1}{2}\log_3 x$, so $3^{\log_9 x} = 3^{\frac{1}{2}\log_3 x} = \left(3^{\log_3 x}\right)^{1/2} = \sqrt{x}$.

c) We switch every logarithm to the natural logarithm. The product telescopes:

$$\frac{\ln 4}{\ln 3} \cdot \frac{\ln 5}{\ln 4} \cdot \frac{\ln 6}{\ln 5} \cdot \frac{\ln 7}{\ln 6} \cdot \frac{\ln 8}{\ln 7} \cdot \frac{\ln 9}{\ln 8} = \frac{\ln 9}{\ln 3} = \frac{\ln\left(3^2\right)}{\ln 3} = \frac{2 \ln 3}{\ln 3} = 2$$

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In summary, suppose that $a, b, c > 0$, $a, c \neq 1$ and x is any real number. Then

1. $\log_a(a^x) = x$
2. $a^{\log_a b} = b$
3. $\log_a b + \log_a c = \log_a(bc)$
4. $\log_a b - \log_a c = \log_a\left(\frac{b}{c}\right)$
5. $x \log_a b = \log_a(b^x)$
6. $\log_a b = \frac{\log_c b}{\log_c a}$

Rules 3, 4, and 5 hold when both sides exist.

Exponential equations with different bases

Example 7. Solve the equation $3 \cdot 5^{2x-3} = 6 \cdot 8^{x-2}$.

Solution: We show two ways.

First way: rules of exponents. We write $5^{2x-3} = \frac{5^{2x}}{5^3} = \frac{25^x}{125}$ and $8^{x-2} = \frac{8^x}{8^2} = \frac{8^x}{64}$.

$$\begin{aligned} \frac{3}{125} \cdot 25^x &= \frac{6}{64} \cdot 8^x && \text{divide by } 8^x, \text{ multiply by } \frac{125}{3} \\ \frac{25^x}{8^x} &= \frac{6 \cdot 125}{64 \cdot 3} \\ \left(\frac{25}{8}\right)^x &= \frac{125}{32} && \text{re-write as a logarithm} \\ x &= \log_{25/8} \frac{125}{32} \end{aligned}$$

Second way: take the natural logarithm of both sides. First we divide by 3 and write everything as powers of primes: $2 \cdot 8^{x-2} = 2 \cdot 2^{3x-6} = 2^{3x-5}$.

$$\begin{aligned} 5^{2x-3} &= 2^{3x-5} && \text{take the natural logarithm of both sides} \\ (2x-3)\ln 5 &= (3x-5)\ln 2 && \text{distribute} \\ 2x\ln 5 - 3\ln 5 &= 3x\ln 2 - 5\ln 2 && \text{collect the } x \text{ terms on the left} \\ x(2\ln 5 - 3\ln 2) &= 3\ln 5 - 5\ln 2 && \text{divide} \\ x &= \frac{3\ln 5 - 5\ln 2}{2\ln 5 - 3\ln 2} \approx 1.1958 \end{aligned}$$

Example 8. Are the two answers in the previous example the same?

Solution: They look very different, but they are equal. We use the change-base theorem to switch to the natural logarithm, and then the quotient rule and the power rule:

$$\log_{25/8} \frac{125}{32} = \frac{\ln \frac{125}{32}}{\ln \frac{25}{8}} = \frac{\ln 125 - \ln 32}{\ln 25 - \ln 8} = \frac{\ln\left(5^3\right) - \ln\left(2^5\right)}{\ln\left(5^2\right) - \ln\left(2^3\right)} = \frac{3\ln 5 - 5\ln 2}{2\ln 5 - 3\ln 2}$$

When we solve exponential equations, correct answers can look very different. The rules of logarithms help us decide whether two answers are the same number.

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Using the rules together

Example 9. Simplify each of the following expressions.

$$\text{a)}\ \log 21 - \frac{1}{2}\log 28 - \log 15 - \log\sqrt{700} \qquad \text{b)}\ 2\log_2\left(2x^5\right) - \log_4\left(144x^8\right) + \frac{1}{3}\log_2\left(216x^6\right)$$

Solution: a) Recall that $\log 21$ is the same as $\log_{10} 21$. By the power rule, $\frac{1}{2}\log 28 = \log\sqrt{28}$.

$$\log 21 - \left(\log\sqrt{28} + \log 15 + \log\sqrt{700}\right) = \log\frac{21}{\sqrt{28}\cdot 15\cdot\sqrt{700}}$$

$$= \log\frac{21}{2\sqrt{7}\cdot 15\cdot 10\sqrt{7}} = \log\frac{3\cdot 7}{2\cdot 7\cdot 15\cdot 10} = \log\frac{1}{100} = -2$$

b) We can only combine logarithms of the same base. We change the second expression to base 2, and we use the power rule to get rid of the coefficients:

$$2\log_2\left(2x^5\right) = \log_2\left(4x^{10}\right), \quad \log_4\left(144x^8\right) = \frac{\log_2\left(144x^8\right)}{\log_2 4} = \frac{1}{2}\log_2\left(144x^8\right) = \log_2\left(12x^4\right)$$

$$\frac{1}{3}\log_2\left(216x^6\right) = \log_2\left(6x^2\right)$$

$$\log_2\left(4x^{10}\right) - \log_2\left(12x^4\right) + \log_2\left(6x^2\right) = \log_2\frac{4x^{10}\cdot 6x^2}{12x^4} = \log_2\left(2x^8\right) = \log_2 2 + \log_2\left(x^8\right) = 1 + 8\log_2 x$$

Example 10. Which of the following is NOT equivalent to $\log_8\left(\frac{50}{3}\right)$?

$$\text{A)}\ \frac{\ln\left(\frac{50}{3}\right)}{\ln 8} \qquad \text{B)}\ \frac{\ln 50 - \ln 3}{\ln 8} \qquad \text{C)}\ \frac{\ln 50 - \ln 3}{3\ln 2} \qquad \text{D)}\ \frac{2\ln 5 + \ln 2 - \ln 3}{3\ln 2} \qquad \text{E)}\ \frac{2\ln 5 - \ln 3}{3}$$

Solution: By the change-base theorem, $\log_8\left(\frac{50}{3}\right) = \frac{\ln\left(\frac{50}{3}\right)}{\ln 8}$, which is choice A. The quotient rule in the numerator gives choice B. Since $\ln 8 = \ln\left(2^3\right) = 3\ln 2$, we get choice C. Since $\ln 50 = \ln\left(2\cdot 5^2\right) = \ln 2 + 2\ln 5$, we get choice D. Choice E is the only one left, and indeed it represents a serious algebraic error: $\frac{a+b-c}{3b} \neq \frac{a-c}{3}$. The answer is E. (We can also check with a calculator: A, B, C, D all give the same decimal, E does not.)

Example 11. Let $x = \log_3 2$. Express each of the following in terms of x.

$$\text{a)}\ \log_3 6 \qquad \text{b)}\ \log_3 12 \qquad \text{c)}\ \log_2 3 \qquad \text{d)}\ \log_{12} 24$$

Solution: a) $\log_3 6 = \log_3(3\cdot 2) = \log_3 3 + \log_3 2 = 1 + x$

b) $\log_3 12 = \log_3\left(3\cdot 2^2\right) = \log_3 3 + 2\log_3 2 = 1 + 2x$

c) If we swap the two numbers in a logarithm, we get the reciprocal: $\log_2 3 = \frac{1}{\log_3 2} = \frac{1}{x}$

d) We switch to base 3. Since $\log_3 24 = \log_3\left(3\cdot 2^3\right) = 1 + 3x$, $\log_{12} 24 = \frac{\log_3 24}{\log_3 12} = \frac{1+3x}{1+2x}$

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Example 12. a) Suppose that $\log_2 6 = a$ and $\log_8 5 = b$. Express $\log_{10} 144$ in terms of a and b.

b) Let $a = \log_3 75$ and $b = \log_2 27$. Express $\log_3 10$ in terms of a and b.

Solution: a) We only need $\log_2 3$ and $\log_2 5$ in terms of a and b. Since $a = \log_2(2 \cdot 3) = 1 + \log_2 3$, we have $\log_2 3 = a - 1$. Since $b = \log_8 5 = \dfrac{\log_2 5}{\log_2 8} = \dfrac{\log_2 5}{3}$, we have $\log_2 5 = 3b$.

$$\log_{10} 144 = \frac{\log_2 144}{\log_2 10} = \frac{\log_2\left(2^4 \cdot 3^2\right)}{\log_2(2 \cdot 5)} = \frac{4 + 2\log_2 3}{1 + \log_2 5} = \frac{4 + 2(a-1)}{1 + 3b} = \frac{2a+2}{3b+1}$$

b) $a = \log_3\left(3 \cdot 5^2\right) = 1 + 2\log_3 5$, so $\log_3 5 = \dfrac{a-1}{2}$. $b = \log_2 27 = \dfrac{\log_3 27}{\log_3 2} = \dfrac{3}{\log_3 2}$, so $\log_3 2 = \dfrac{3}{b}$. Therefore

$$\log_3 10 = \log_3 2 + \log_3 5 = \frac{3}{b} + \frac{a-1}{2} = \frac{ab - b + 6}{2b}$$

Example 13. a) Simplify $\dfrac{\log_3 90}{\log_{30} 3} - \dfrac{\log_3 270}{\log_{10} 3}$. b) Write $\log_2 5 - \log_4 10$ as a single logarithm.

Solution: a) Swapping the two numbers in a logarithm gives the reciprocal, so dividing by $\log_{30} 3$ is the same as multiplying by $\log_3 30$. Let $L = \log_3 10$. Then $\log_3 90 = 2 + L$, $\log_3 30 = 1 + L$, and $\log_3 270 = 3 + L$, so

$$\frac{\log_3 90}{\log_{30} 3} - \frac{\log_3 270}{\log_{10} 3} = (2+L)(1+L) - (3+L)L = 2 + 3L + L^2 - 3L - L^2 = 2$$

b) The two logarithms have different bases. We show two ways.

First way: everything in base 2. By the change-base theorem, $\log_4 10 = \dfrac{\log_2 10}{\log_2 4} = \dfrac{1}{2}\log_2 10 = \log_2 \sqrt{10}$, so

$$\log_2 5 - \log_4 10 = \log_2 5 - \log_2 \sqrt{10} = \log_2 \frac{5}{\sqrt{10}} = \log_2 \frac{\sqrt{10}}{2}$$

Second way: everything in base 4. If we square both the base and the number, the logarithm does not change: $\log_2 5 = \log_4 25$, so

$$\log_2 5 - \log_4 10 = \log_4 25 - \log_4 10 = \log_4 \frac{25}{10} = \log_4 \frac{5}{2}$$

Both answers are correct, and they are equal.

Domains

The domain of an expression is the set of all numbers we can substitute for x. For logarithms, every argument must be positive (and every base must be positive and different from 1).

Example 14. Find the domain of a) $\log_3\left(x^2 - 16\right)$ b) $\log_3(x+4) + \log_3(x-4)$

Solution: a) We need $x^2 - 16 > 0$, that is, $(x+4)(x-4) > 0$. This happens if $x < -4$ or $x > 4$. In interval notation: $(-\infty, -4) \cup (4, \infty)$.

b) Here both arguments must be positive: $x + 4 > 0$ and $x - 4 > 0$, so $x > 4$. In interval notation: $(4, \infty)$.

Notice that by the product rule, the two expressions look the same, but their domains are different. This is what 'when both sides exist' means in the product rule.

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Example 15. Find the domain of a) $\dfrac{1}{\ln(x-3)}$ b) $\dfrac{1}{\log_3(2x-1)-4}$ c) $\dfrac{3}{\log_{10}(2x-x^2)}$

Solution: a) We need $x-3>0$, and the denominator cannot be zero: $\ln(x-3)=0$ exactly when $x-3=1$. So $x>3$, $x\neq 4$. In interval notation: $(3,\infty)\setminus\{4\}$.

b) We need $2x-1>0$, that is, $x>\dfrac{1}{2}$, and $\log_3(2x-1)\neq 4$, that is, $2x-1\neq 81$, $x\neq 41$. In interval notation: $\left(\dfrac{1}{2},\infty\right)\setminus\{41\}$.

c) We need $2x-x^2=x(2-x)>0$, which happens if $0<x<2$, and $\log_{10}\left(2x-x^2\right)\neq 0$, that is, $2x-x^2\neq 1$. Since $2x-x^2=1$ means $(x-1)^2=0$, we exclude $x=1$. In interval notation: $(0,2)\setminus\{1\}$.

Logarithmic equations

In a logarithmic equation, we use the rules of logarithms to get to a single logarithm, and then we re-write the logarithmic statement as an exponential statement. Using the rules can change the domain, so we must ALWAYS check our solutions in the original equation.

Type 1: a logarithm equals a number

Example 16. Solve each of the following equations. a) $\log_2[(x-3)(x+1)]=5$ b) $\log_2(x-3)+\log_2(x+1)=5$

Solution: a) We re-write the logarithmic statement as an exponential statement.

$$\begin{aligned}(x-3)(x+1) &= 2^5=32 && \text{multiply out}\\ x^2-2x-3 &= 32 && \text{subtract 32}\\ x^2-2x-35 &= 0 && \text{factor}\\ (x+5)(x-7) &= 0 && x_1=-5 \text{ and } x_2=7\end{aligned}$$

We check: if $x=-5$, then LHS $=\log_2[(-8)(-4)]=\log_2 32=5$. If $x=7$, then LHS $=\log_2[(4)(8)]=\log_2 32=5$. Both are solutions.

b) By the product rule, this equation appears to be identical to the previous one. But it is not. Let's check: if $x=-5$, then LHS $=\log_2(-8)+\log_2(-4)$, which is undefined. If $x=7$, then LHS $=\log_2 4+\log_2 8=2+3=5$. So this equation has only one solution, $x=7$. It is very important to check.

Example 17. Solve each of the following equations. a) $\log_2(x+29)-\log_2(x-3)=1$ b) $\log_2(x-3)-\log_2(x+1)=1$

Solution: a) By the quotient rule, $\log_2\dfrac{x+29}{x-3}=1$, so $\dfrac{x+29}{x-3}=2$. We multiply by $x-3$, and the equation becomes linear: $x+29=2x-6$, $x=35$. Check: LHS $=\log_2 64-\log_2 32=6-5=1$. ✓

b) In the same way, $\dfrac{x-3}{x+1}=2$, $x-3=2x+2$, $x=-5$. But then $x-3=-8$, and $\log_2(-8)$ is not defined. This equation has no solution.

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Type 2: a logarithm equals a logarithm

Two logarithms with the same base are equal only if their arguments are equal. (If $\log_a A = \log_a B$, then $a^{\log_a A} = a^{\log_a B}$, that is, $A = B$.)

Example 18. Solve each of the following equations. a) $\log_3\left(x^2 - 4x\right) = \log_3(2x - 5)$ b) $2\log_7 x = \log_7(x + 6)$

Solution: a) The arguments are equal: $x^2 - 4x = 2x - 5$, $x^2 - 6x + 5 = 0$, $(x - 1)(x - 5) = 0$. If $x = 1$, then $2x - 5 = -3$, and $\log_3(-3)$ is not defined, so $x = 1$ is NOT a solution. If $x = 5$, both arguments are 5. The only solution is $x = 5$.

b) By the power rule, $2\log_7 x = \log_7\left(x^2\right)$, so $x^2 = x + 6$, $x^2 - x - 6 = 0$, $(x - 3)(x + 2) = 0$. $x = -2$ is NOT a solution, because $\log_7(-2)$ is not defined. If $x = 3$, then LHS $= 2\log_7 3 = \log_7 9 =$ RHS. The only solution is $x = 3$.

Type 3: the power rule and the domain

Example 19. a) Solve the equation $2\log_5 x - \log_5(x + 30) = 1$.

b) Solve the equations $\log_3\left(x^2\right) = 4$ and $2\log_3 x = 4$. Why are the solutions different?

Solution: a) We use the power rule and then the quotient rule to get a single logarithm.

$$\begin{aligned} \log_5 \frac{x^2}{x+30} &= 1 \quad \text{re-write as exponential} \\ \frac{x^2}{x+30} &= 5 \quad \text{multiply by } x+30 \\ x^2 - 5x - 150 &= 0 \quad \text{factor} \\ (x-15)(x+10) &= 0 \quad x_1 = 15 \text{ and } x_2 = -10 \end{aligned}$$

Since $\log_5 x$ is defined only for $x > 0$, $x = -10$ is NOT a solution. If $x = 15$, then LHS $= \log_5 \dfrac{225}{45} = \log_5 5 = 1$. The only solution is $x = 15$.

b) $\log_3\left(x^2\right) = 4$ means $x^2 = 81$, so $x = \pm 9$, and both are solutions. But $2\log_3 x = 4$ means $\log_3 x = 2$, so $x = 9$ only: $\log_3(-9)$ is not defined. The two equations look the same by the power rule, but the power rule only holds when both sides exist: $\log_3\left(x^2\right)$ is defined for every $x \neq 0$, while $2\log_3 x$ is defined only for $x > 0$.

Type 4: the natural logarithm

Example 20. Solve each of the following equations. Give exact values. a) $\ln(2x + 1) = 3$ b) $\ln x + \ln(x + 3) = \ln 10$ c) $2\ln x = \ln(x + 12)$

Solution: a) Recall that ln is the logarithm with base e. So $2x + 1 = e^3$, and $x = \dfrac{e^3 - 1}{2}$. Since $2x + 1 = e^3 > 0$, this is indeed a solution.

b) By the product rule, $\ln[x(x + 3)] = \ln 10$, so $x^2 + 3x = 10$, $(x + 5)(x - 2) = 0$. $x = -5$ is NOT a solution ($\ln(-5)$ is not defined). The only solution is $x = 2$: $\ln 2 + \ln 5 = \ln 10$. ✓

c) $\ln\left(x^2\right) = \ln(x + 12)$, so $x^2 - x - 12 = 0$, $(x - 4)(x + 3) = 0$. $x = -3$ is NOT a solution. The only solution is $x = 4$: $2\ln 4 = \ln 16 = \ln(4 + 12)$. ✓

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Type 5: different bases

Example 21. Solve each of the following equations. a) $\log_2 x + \log_8 x = 4$ b) $\log_4 x - \log_{16} x = 1$ c) $\log_{64} x + \log_x 64 = \dfrac{13}{6}$

Solution: a) By the change-base theorem, $\log_8 x = \dfrac{\log_2 x}{\log_2 8} = \dfrac{1}{3}\log_2 x$. So $\dfrac{4}{3}\log_2 x = 4$, $\log_2 x = 3$, $x = 2^3 = 8$. Check: $\log_2 8 + \log_8 8 = 3 + 1 = 4$. ✓

b) $\log_{16} x = \dfrac{\log_4 x}{\log_4 16} = \dfrac{1}{2}\log_4 x$, so $\dfrac{1}{2}\log_4 x = 1$, $\log_4 x = 2$, $x = 16$. Check: $\log_4 16 - \log_{16} 16 = 2 - 1 = 1$. ✓

c) Swapping the two numbers in a logarithm gives the reciprocal: $\log_x 64 = \dfrac{1}{\log_{64} x}$. Let $u = \log_{64} x$. Then $u + \dfrac{1}{u} = \dfrac{13}{6}$. We multiply by $6u$: $6u^2 - 13u + 6 = 0$, $(2u - 3)(3u - 2) = 0$, so $u = \dfrac{3}{2}$ or $u = \dfrac{2}{3}$. Then $x = 64^{3/2} = 512$ or $x = 64^{2/3} = 16$. Both are solutions.

Type 6: substitution

Example 22. Solve each of the following equations.

a) $(\log_2 x)^2 - 5\log_2 x + 6 = 0$

b) $\log_2(\log_5 x) = 2$

c) $\sqrt{\log x} = \log\sqrt{x}$

d) $\log\sqrt{x} = \sqrt{\log x - 1}$

e) $\log\sqrt{x} = \sqrt{\log x + 3}$

Solution: a) The equation is quadratic in $\log_2 x$. Let $u = \log_2 x$: $u^2 - 5u + 6 = 0$, $(u - 2)(u - 3) = 0$. If $\log_2 x = 2$, then $x = 4$; if $\log_2 x = 3$, then $x = 8$. The solutions are $x = 4$ and $x = 8$.

b) We re-write the logarithmic statement as an exponential statement twice, from the outside in: $\log_5 x = 2^2 = 4$, so $x = 5^4 = 625$.

c) By the power rule, $\log\sqrt{x} = \dfrac{1}{2}\log x$. Let $u = \log x$ (because of the square root, $u \geq 0$). Then $\sqrt{u} = \dfrac{u}{2}$. We square both sides: $u = \dfrac{u^2}{4}$, $u^2 - 4u = 0$, and we factor out the GCF: $u(u - 4) = 0$. Squaring can create extra solutions, so we check: $\sqrt{0} = \dfrac{0}{2}$ and $\sqrt{4} = \dfrac{4}{2}$. Both work. So $\log x = 0$ or $\log x = 4$, that is, $x = 1$ or $x = 10\,000$.

d) With $u = \log x$: $\dfrac{u}{2} = \sqrt{u - 1}$, $\dfrac{u^2}{4} = u - 1$, $u^2 - 4u + 4 = 0$, $(u - 2)^2 = 0$, $u = 2$. Check: $\dfrac{2}{2} = 1 = \sqrt{2 - 1}$. ✓ So $x = 10^2 = 100$.

e) With $u = \log x$: $\dfrac{u}{2} = \sqrt{u + 3}$, $u^2 - 4u - 12 = 0$, $(u - 6)(u + 2) = 0$. If $u = 6$: $\dfrac{6}{2} = 3 = \sqrt{9}$. ✓ If $u = -2$, the left-hand side is -1, which is negative, but a square root is never negative. So $u = -2$ is an extra solution created by squaring, and we reject it. The only solution is $x = 10^6 = 1\,000\,000$.

Applications

Example 23. The magnitude of an earthquake on the Richter scale is $M = \log\dfrac{A}{A_0}$, where A is the amplitude of the earthquake's wave measured by a seismograph, and A_0 is a small reference amplitude.

a) The amplitude of an earthquake is $50\,000$ times the reference amplitude. What is its magnitude?

b) How many times as large is the amplitude of a magnitude 7.3 earthquake as the amplitude of a magnitude 5.3 earthquake?

Solution: a) We have $A = 50\,000A_0$, so $M = \log\dfrac{50\,000A_0}{A_0} = \log 50\,000 = \log\left(5 \cdot 10^4\right) = 4 + \log 5 \approx 4.7$.

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b) For the stronger earthquake, $7.3 = \log \dfrac{A_1}{A_0}$, so $A_1 = 10^{7.3} A_0$. For the weaker earthquake, in the same way, $A_2 = 10^{5.3} A_0$. We divide; A_0 cancels:

$$\frac{A_1}{A_2} = \frac{10^{7.3} A_0}{10^{5.3} A_0} = 10^{7.3-5.3} = 10^2 = 100$$

The amplitude of the stronger earthquake is 100 times as large. Each step of 1 on the Richter scale means a factor of 10 in the amplitude.

Practice Problems

1. Simplify each of the following expressions.

   a) $\log_{10} 5 + \log_{10} 2$
   b) $\log_2 (40a) - \log_2 (5a)$
   c) $\log_{10} 0.0002 + \log_{10} 5$
   d) $4^{\log_2 a}$
   e) $2^{\log_4 y}$
   f) $2 \ln \sqrt{x^2 - 1} - \ln (x + 1)$
   g) $\log_6 \sqrt{12} + \log_6 \sqrt{18}$
   h) $\log_5 (3x) + \log_5 \left(15x^2\right) - 2 \log_5 3$
   i) $2 \ln \sqrt{m} + 3 \ln \sqrt[3]{m}$
   j) $2 \log_3 \left(2A^5\right) - \log_9 \left(144A^8\right)$
   k) $\log_3 \left(12b^2\right) - 2 \log_3 (2b)$
   l) $\log_{\sqrt{m}} \sqrt[3]{m^7}$
   m) $\log \sqrt{52} + 3 \log 2 + \log 125 + \log \sqrt{325} - \log 13$
   n) $\log_a \left(\left(3 - \dfrac{3a - 2}{a + 1} \right) \cdot \dfrac{a^2 + a}{5} \right)$

2. Which of the following is NOT equivalent to $\log_9 \left(\dfrac{36}{25}\right)$?

   A) $\dfrac{\ln \left(\frac{36}{25}\right)}{\ln 9}$ B) $\dfrac{\ln \left(\frac{6}{5}\right)}{2 \ln 3}$ C) $\dfrac{\ln 36 - \ln 25}{2 \ln 3}$ D) $\dfrac{\ln 36 - \ln 25}{\ln 9}$ E) $\dfrac{\ln 6 - \ln 5}{\ln 3}$

3. Let $x = \log_3 2$. Express each of the following in terms of x.

   a) $\log_3 18$ b) $\log_3 24$ c) $\log_3 72$ d) $\log_3 \left(\dfrac{2}{3}\right)$ e) $\log_3 \left(\dfrac{9}{8}\right)$ f) $\log_{72} 24$

4. Let $x = \log_2 5$. Express each of the following in terms of x.

   a) $\log_2 125$
   b) $\log_2 10$
   c) $\log_2 1000$
   d) $\log_2 80$
   e) $\log_2 \left(\dfrac{5}{2}\right)$
   f) $\log_5 2$
   g) $\log_{10} 2$
   h) $\log_5 10$
   i) $\log_{50} 80$
   j) $\log_{100} 10$
   k) $\log_2 \left(\dfrac{16}{25}\right)$
   l) $\log_5 \left(\dfrac{16}{25}\right)$

5. Let $p = \log_2 5$ and $q = \log_5 3$. Express each of the following in terms of p and q.

   a) $\log_2 10$ b) $\log_5 45$ c) $\log_5 10$ d) $\log_5 6$ e) $\log_5 30$ f) $\log_2 3$ g) $\log_2 24$ h) $\log_{24} 30$

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6. a) Solve the equation $3^{2x-1} = 2^{x+3}$ in two ways: first using the rules of exponents, and then by taking the natural logarithm of both sides. Show that the two answers are equal.
   b) Show that $\log_{8/15}\left(\frac{24}{25}\right) = \frac{3\ln 2 + \ln 3 - 2\ln 5}{3\ln 2 - \ln 3 - \ln 5}$.

7. Find the domain of each of the following expressions.

   a) $\log_5\left(-x^2 + 10x - 23\right)$
   b) $\log_2\left(x^2 - 6x + 8\right)$
   c) $\dfrac{1}{\log_2\left(x^2 - 6x + 8\right)}$
   d) $\dfrac{1}{\log_2(x-2) + \log_2(x-4)}$
   e) $\ln\left(\dfrac{x^2 - 6x}{4 - x^2}\right)$
   f) $\ln\left(x^2 - 6x\right) - \ln\left(4 - x^2\right)$
   g) $\dfrac{\ln\left(x^2 - 6x\right)}{\ln\left(4 - x^2\right)}$
   h) $\log_3\left(x^2 + 1\right)$

8. Solve each of the following equations.

   a) $\log_6(8 - x) + \log_6(x + 12) = 2$
   b) $\log_4(3m + 5) - \log_4(m + 7) = \dfrac{1}{2}$
   c) $\log_2(3x - 5) + \log_2(x - 6) = 4$
   d) $\log_2(2 - y) + \log_2(10 - y) = 7$
   e) $\log_x(12 - x) = 2$
   f) $\log_{x-1}(x + 2) + \log_{x-1}(x - 2) = 2$
   g) $\log_2 x + \log_2(x - 4) = 5$
   h) $\log_4(x - 1) + \log_4(x + 3) = \dfrac{5}{2}$
   i) $\log_6 x + \log_6(2x + 1) = 2$
   j) $\log_2(x - 5) + \log_2(x + 11) = 9$
   k) $\log_6 2 + \log_6(2x - 5) + \log_6(x - 5) = 2$

9. Solve each of the following equations.

   a) $\log_5\left(x^2 - 3x\right) = \log_5(x + 5)$
   b) $2\log_3 x = \log_3(x + 12)$

10. a) Solve the equation $2\log_3 x - \log_3(x + 6) = 1$.
    b) Solve the equations $\log_5\left(x^2\right) = 2$ and $2\log_5 x = 2$.

11. Solve each of the following equations. Give exact values.

    a) $\ln(3x - 2) = 2$
    b) $\ln x + \ln(x - 2) = \ln 15$
    c) $2\ln x = \ln(2x + 15)$

12. Solve each of the following equations.

    a) $\log_3 x + \log_9 x = 3$
    b) $\log_2 x - \log_4 x = 2$
    c) $\log_5 x + \log_{25} x + \log_{125} x = \dfrac{11}{3}$

13. Solve each of the following equations.

    a) $(\log_3 x)^2 - \log_3 x - 2 = 0$
    b) $\log_3(\log_2 x) = 1$
    c) $\sqrt{\log_2 x} = \log_2 \sqrt{x}$
    d) $\log_3 \sqrt[3]{x} = \sqrt{\log_3 x - 2}$
    e) $\log \sqrt{x} = \sqrt{\log x + 8}$

14. (Applications) a) The amplitude of an earthquake is 2000 times the reference amplitude. What is its magnitude on the Richter scale?
    b) How many times as large is the amplitude of a magnitude 6.8 earthquake as the amplitude of a magnitude 3.8 earthquake?

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Enrichment

15. Prove that $\log_{a/b}\left(\frac{c}{d}\right) = \log_{b/a}\left(\frac{d}{c}\right)$. (Assume that $a, b, c, d > 0$ and $a \neq b$.)

16. Solve each of the following equations.

a) $\left[64^{\frac{2}{3}} \cdot 3^{-\log_{27} 8}\right]^{\frac{1}{3}} + \log_2\left(x^3\right) = 14$

b) $\log_{2x} 16 + \log_{4x} 8 = \log_x 8$

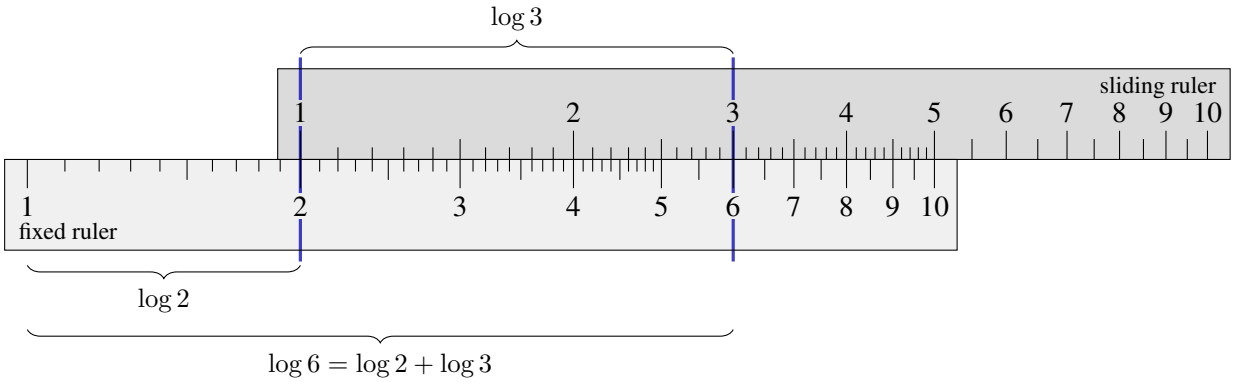
c) $x\left(1 - \log_{21} 3\right) = \log_{21} 30 - \log_{21}\left(7^x + 1\right)$

d) $\log_x(x-3) \cdot \log_{x-3}(x+20) = 2$

e) $\dfrac{\log_2 x \cdot \log_3 x}{\log_2 x + \log_3 x} = 1$

The slide rule. Before electronic calculators became common in the 1970s, engineers and scientists multiplied numbers with a **slide rule**. It is made of two rulers that slide along each other. Each ruler has a logarithmic scale: the number x is marked at distance $\log x$ from the left end (measured in the length of the whole ruler). So the left end is marked 1, because $\log 1 = 0$, and the right end is marked 10, because $\log 10 = 1$.

The picture shows how to compute $2 \cdot 3$. We slide the 1 of the sliding ruler over the 2 of the fixed ruler. Then we find 3 on the sliding ruler, and we read the number under it on the fixed ruler: 6. The two distances $\log 2$ and $\log 3$ are placed end to end, so together they are $\log 2 + \log 3 = \log(2 \cdot 3) = \log 6$. The product rule does the multiplication for us!



17. a) Use the picture to compute $2 \cdot 4$.
    b) How can we use the slide rule to divide 6 by 3? Which rule of logarithms makes it work?
    c) Why is there no 0 on the scales of a slide rule?
    d) Try to compute $4 \cdot 5$ the same way. What goes wrong? How can we fix it? (Hint: place the 10 of the sliding ruler over the 4 of the fixed ruler.)

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Answers

1. a) 1 b) 3 c) -3 d) a^2 e) $\sqrt{y}$ f) $\ln(x-1)$ g) $\frac{3}{2}$ h) $1+3\log_5 x$ i) $2\ln m$ j) $6\log_3 A - 1$ k) 1 l) $\frac{14}{3}$ m) 4 n) 1
2. B
3. a) $x+2$ b) $3x+1$ c) $3x+2$ d) $x-1$ e) $2-3x$ f) $\frac{3x+1}{3x+2}$
4. a) $3x$ b) $x+1$ c) $3x+3$ d) $x+4$ e) $x-1$ f) $\frac{1}{x}$ g) $\frac{1}{x+1}$ h) $\frac{x+1}{x}$ i) $\frac{x+4}{2x+1}$ j) $\frac{1}{2}$ k) $4-2x$ l) $\frac{4-2x}{x}$
5. a) $p+1$ b) $2q+1$ c) $\frac{1}{p}+1$ d) $\frac{1}{p}+q$ e) $\frac{1}{p}+q+1$ f) pq g) $pq+3$ h) $\frac{p+pq+1}{pq+3}$
6. a) Rules of exponents: $\frac{1}{3}\cdot 9^x = 8\cdot 2^x$, $\left(\frac{9}{2}\right)^x = 24$, $x = \log_{9/2} 24$. Natural logarithm: $(2x-1)\ln 3 = (x+3)\ln 2$, $x = \frac{\ln 3 + 3\ln 2}{2\ln 3 - \ln 2} \approx 2.1130$. They are equal, because $\log_{9/2} 24 = \frac{\ln 24}{\ln\left(\frac{9}{2}\right)} = \frac{\ln 3 + 3\ln 2}{2\ln 3 - \ln 2}$.
   b) $\log_{8/15}\left(\frac{24}{25}\right) = \frac{\ln 24 - \ln 25}{\ln 8 - \ln 15}$, and $\ln 24 = 3\ln 2 + \ln 3$, $\ln 25 = 2\ln 5$, $\ln 8 = 3\ln 2$, $\ln 15 = \ln 3 + \ln 5$.
7. a) $\left(5-\sqrt{2}, 5+\sqrt{2}\right)$ b) $(-\infty, 2)\cup(4,\infty)$ c) $(-\infty,2)\cup(4,\infty)\setminus\{3-\sqrt{2}, 3+\sqrt{2}\}$ d) $(4,\infty)\setminus\{3+\sqrt{2}\}$ e) $(-2,0)\cup(2,6)$ f) $(-2,0)$ g) $(-2,0)\setminus\{-\sqrt{3}\}$ h) all real numbers
8. a) $-10,\ 6$ b) 9 c) 7 d) -6 e) 3 f) $\frac{5}{2}$ g) 8 h) 5 i) 4 j) 21 k) 7
9. a) $-1,\ 5$ b) 4
10. a) 6 b) $\log_5(x^2) = 2$: $x = \pm 5$; $2\log_5 x = 2$: $x = 5$
11. a) $\frac{e^2+2}{3}$ b) 5 c) 5
12. a) 9 b) 16 c) 25
13. a) $\frac{1}{3},\ 9$ b) 8 c) $1,\ 16$ d) $27,\ 729$ e) 10^8
14. a) $\log 2000 = 3 + \log 2 \approx 3.3$ b) $10^3 = 1000$ times as large

Enrichment

15. Hint: switch to the natural logarithm. $\ln\left(\frac{c}{d}\right) = -\ln\left(\frac{d}{c}\right)$ and $\ln\left(\frac{a}{b}\right) = -\ln\left(\frac{b}{a}\right)$, so the two negative signs cancel.
16. a) 16 b) $\frac{1}{\sqrt{8}},\ 2$ c) $\log_7 5$ d) 5 e) 6
17. a) 8 b) 3 over 6, read 2 under the 1: $\log 6 - \log 3 = \log 2$ c) $\log 0$ is not defined
    d) $\log 4 + \log 5 = \log 20 > 1$, past the end; with 10 over 4, read 2 under 5, so $4\cdot 5 = 20$

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