

This is the second part of studying logarithms. Before we do that, let us recall a few things from the first part.

Definition: The symbol $\log_2 8$ represents the number of 2-factors needed to multiply each other, so that the resulting product is 8.

Therefore, $\log_2 8$ is 3. Indeed, we need three 2-factors multiplying each other so that the result is 8. In short,

$$\log_2 8 = 3 \quad \text{because} \quad 2^3 = 8$$

So, in a sense, logarithms are exponents. We need 3 on the top of 2 as an exponent for a product of 8. We can express this concept more formally. The symbol $\log_a b$ represents the number of a -factors needed to multiply each other, so that the resulting product is b .

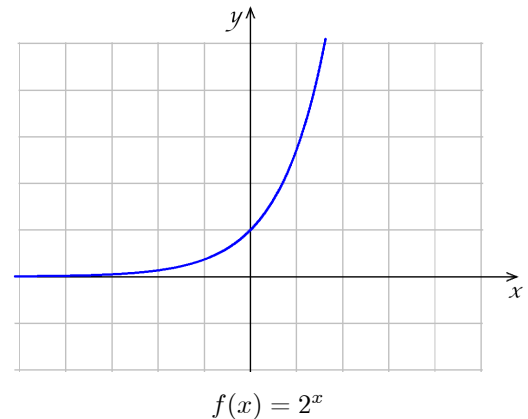
$$\log_a b = x \quad \text{because} \quad a^x = b$$

Every logarithmic statement can be re-written as an exponential statement. The number a is called the **base** of the logarithm - it is also the base of the corresponding exponential statement.

When we extended the concept of exponents from the integers to the real numbers, we ran into trouble with negative bases. So, the base of the logarithm must be positive, and cannot be 1. (This is because every power of 1 is 1.)

Considering the exponential function $f(x) = 2^x$, we noticed that 2^x has only positive values. In other words, 0 or negative numbers cannot be achieved as 2-powers. Therefore, $\log_a b$ is only defined if b is positive.

In summary,



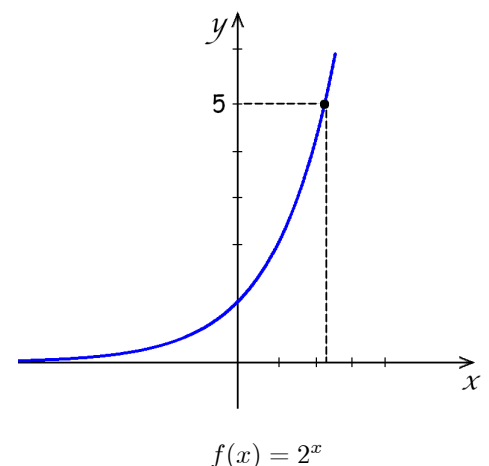
Definition: The symbol $\log_a b$ represents the number of a -factors needed to multiply each other, so that the resulting product is b .

$$\log_a b = x \quad \text{because} \quad a^x = b$$

where $a, b > 0$ and $a \neq 1$

Naturally, we did not need the concept or notation of logarithms for a number like $\log_2 8$. We needed them for numbers such as $\log_2 5$.

Because $f(x) = 2^x$ is a continuous function, it must take the y -value 5 somewhere. The x -coordinate of that point is $\log_2 5$. This is an irrational number, and there is no other way to represent its exact value.



We saw two theorems as immediate consequences of the definition:

Theorem 1. If $a, b > 0$, $a \neq 1$, and k any real number, then $\log_a (a^k) = k$.

Theorem 2. If $a, b > 0$, $a \neq 1$, then $a^{\log_a b} = b$.

For the next theorem, the 'ingredients' are the first rule of exponentiation, $a^n \cdot a^m = a^{n+m}$, and the fact that $a^x = a^y$ implies $x = y$ for all $a > 0$, x, y real numbers, because $f(x) = a^x$ is a one-to-one function.

Theorem 3. When both sides exist, then $\log_a x + \log_a y = \log_a xy$.

proof. Consider first $a^{\log_a x + \log_a y}$ and $a^{\log_a xy}$.

Clearly, $a^{\log_a xy} = xy$ by the second theorem. The other expression,

$$a^{\log_a x + \log_a y} = a^{\log_a x} \cdot a^{\log_a y} = xy.$$

Therefore, $a^{\log_a x + \log_a y} = a^{\log_a xy}$. Since $f(x) = a^x$ is a one-to-one function, two a -powers can only be equal if their exponents are the same. Therefore, $\log_a x + \log_a y = \log_a xy$. This completes our proof.



Discussion: Consider the condition 'when both sides exist' in theorem 3. When do both sides exist? Is it possible for only one side to exist, but not the other one?

Theorem 4. When both sides exist, then $\log_a x - \log_a y = \log_a \left(\frac{x}{y} \right)$.

proof. Recall now the second rule of exponents, $\frac{a^n}{a^m} = a^{n-m}$. Consider $a^{\log_a x - \log_a y}$ and $a^{\log_a \left(\frac{x}{y} \right)}$.

Clearly, $a^{\log_a \left(\frac{x}{y} \right)} = \frac{x}{y}$ by the second theorem. The other expression,

$$a^{\log_a x - \log_a y} = \frac{a^{\log_a x}}{a^{\log_a y}} = \frac{x}{y}.$$

Therefore, $a^{\log_a x - \log_a y} = a^{\log_a \left(\frac{x}{y} \right)}$. Since $f(x) = a^x$ is a one-to-one function, two a -powers can only be equal if their exponents are the same. Therefore, $\log_a x - \log_a y = \log_a \left(\frac{x}{y} \right)$.

Theorem 5. When both sides exist, then $\log_a (x^y) = y \log_a x$.

proof. Recall now the third rule of exponents, $(a^n)^m = a^{nm}$. Consider $a^{\log_a (x^y)}$ and $a^{y \log_a x}$.

Clearly, $a^{\log_a (x^y)} = x^y$ by the second theorem. The other expression,

$$a^{y \log_a x} = (a^{\log_a x})^y = x^y.$$

Therefore, $a^{\log_a (x^y)} = a^{y \log_a x}$. Since $f(x) = a^x$ is a one-to-one function, two a -powers can only be equal if their exponents are the same. Therefore, $\log_a (x^y) = y \log_a x$. This completes our proof.

These are very useful properties. For example, Theorem 5. enables us to get approximate values using the calculator. A scientific calculator is usually programmed to compute two logarithms: $\log a$ is short for $\log_{10} a$, called the **common logarithm**, and $\ln a$ is short for $\log_e a$, also called the **natural logarithm**. How could we compute exact values for numbers such as $\log_2 7$? This newly found property enables us to do just that. Let us denote $\log_2 7$ by x . Then

$$\begin{aligned} 2^x &= 7 && \text{let us take the common logarithm of both sides} \\ \log(2^x) &= \log 7 && \text{Use theorem 5 to get exponent to the baseline} \\ x \log 2 &= \log 7 && \text{solve for } x \text{ by dividing both sides by } \log 2 \\ x &= \frac{\log 7}{\log 2} \end{aligned}$$

We can use the calculator with expressions such as $\frac{\log 7}{\log 2} \approx 2.8074$.

The last computation can be performed using any base besides 10. Consider $\log_a b = x$.

$$\begin{aligned} a^x &= b && \text{let us take base } c \text{ logarithm of both sides} \\ \log_c(a^x) &= \log_c b && \text{Use theorem 5 to get exponent to the baseline} \\ x \log_c a &= \log_c b && \text{solve for } x \text{ by dividing both sides by } \log_c a \\ x &= \frac{\log_c b}{\log_c a} \implies \log_a b = \frac{\log_c b}{\log_c a} \end{aligned}$$

This is the proof for our last theorem, the change-base theorem.

Theorem 6. For all $a, b, c > 0$, $a, c \neq 1$, $\log_a b = \frac{\log_c b}{\log_c a}$.

All together, we will use these six theorem to compute with logarithms. There are many other statements, but they can be easily derived from these six.



Enrichment

Prove each of the following. Assume that $a, b > 0$, $a \neq 1$ and n any real number,

1. $\log_{(a^n)}(b^n) = \log_a b$.

2. $\log_b a = \frac{1}{\log_a b}$

In summary, we have the following six theorems.

Suppose that $a, b, c > 0$, $a, c \neq 1$ and x is any real number. Then the following are all true.

1. $\log_a (a^x) = x$

2. $a^{\log_a x} = x$

3. When both sides exist,

$$\log_a b + \log_a c = \log_a bc$$

4. When both sides exist,

$$\log_a b - \log_a c = \log_a \left(\frac{b}{c} \right)$$

5. When both sides exist,

$$x \log_a b = \log_a (b^x)$$

6. The change-base theorem:

$$\log_a b = \frac{\log_c b}{\log_c a}$$



Sample Problems

1. Simplify each of the following expressions.

a) $\log_6 4 + \log_6 54$

b) $1 + 2 \log_2 3 - \log_2 36$

c) $2 \log_{10} (2x) + \log_{10} 25x$

d) $\log 21 - \frac{1}{2} \log 28 - \log 15 - \log \sqrt{700}$

e) $2 \log_2 (2x^5) - \log_4 (144x^8) + \frac{1}{3} \log_2 (216x^6)$

f) $\log_a \left(\left(3 - \frac{3a-2}{a+1} \right) \cdot \frac{a^2+a}{5} \right)$

g) $\log_9 \sqrt{27}$

h) $\log_{\sqrt{m}} \sqrt[3]{m^7}$

i) $e^{-3 \ln 5}$

j) $8^{\log_2 x}$

k) $3^{\log_9 x}$

l) $(\log_3 4)(\log_4 5)(\log_5 6)(\log_6 7)(\log_7 8)(\log_8 9)$

2. Which of the following is NOT equivalent to $\log_8 \left(\frac{50}{3} \right)$?

A) $\frac{\ln \left(\frac{50}{3} \right)}{\ln 8}$

B) $\frac{\ln 50 - \ln 3}{\ln 8}$

C) $\frac{\ln 50 - \ln 3}{3 \ln 2}$

D) $\frac{2 \ln 5 + \ln 2 - \ln 3}{3 \ln 2}$

E) $\frac{2 \ln 5 - \ln 3}{3}$

3. Prove that $\log_{(8/15)} \left(\frac{24}{25} \right) = \frac{3 \ln 2 + \ln 3 - 2 \ln 5}{3 \ln 2 - \ln 3 - \ln 5}$.

4. Let $x = \log_3 2$. Express each of the following in terms of x .

a) $\log_3 6$

c) $\log_3 12$

e) $\log_3 72$

g) $\log_{12} 24$

i) $\log_3 \left(\frac{9}{8} \right)$

b) $\log_3 18$

d) $\log_3 24$

f) $\log_2 3$

h) $\log_3 \left(\frac{2}{3} \right)$

j) $\log_{72} 24$

5. a) Suppose that $\log_2 6 = a$ and $\log_8 5 = b$. Express $\log_{10} 144$ in terms of a and b .
 b) Let $a = \log_3 75$ and $b = \log_2 27$. Express $\log_3 10$ in terms of a and b .
6. a) Simplify $\frac{\log_3 90}{\log_{30} 3} - \frac{\log_3 270}{\log_{10} 3}$ b) Write $\log_2 5 - \log_4 10$ as a single logarithm.
7. a) Prove that $\log_{(a^k)} (b^k) = \log_a b$. b) Prove that $\log_{a/b} \left(\frac{c}{d}\right) = \log_{b/a} \left(\frac{d}{c}\right)$.
8. Find the domain of each of the following expressions.
- a) $\log_3 (x^2 - 16)$ c) $\frac{1}{\ln(x-3)}$ e) $\frac{1}{\log_3(2x-1) - 4}$
 b) $\log_3(x+4) + \log_3(x-4)$ d) $\log_3(x^2 + 1)$ f) $\frac{3}{\log_{10}(2x-x^2)}$
9. Solve each of the following equations.
- a) $\log_2(x-3)(x+1) = 5$ e) $\log_2(x-3) - \log_2(x+1) = 1$
 b) $\log_2(x-3) + \log_2(x+1) = 5$ f) $\left[64^{\frac{2}{3}} \cdot 3^{-\log_{27} 8}\right]^{\frac{1}{3}} + \log_2 x^3 = 14$
 c) $\log_2(x+29) - \log_2(x-3) = 1$ g) $\log_{64} x + \log_x 64 = \frac{13}{6}$
 d) $\log_6 2 + \log_6(2x-5) + \log_6(x-5) = 2$
10. (Enrichment) Solve each of the following equations.
- a) $\log_{2x} 16 + \log_{4x} 8 = \log_x 8$ c) $\log_x(x-3) \cdot \log_{x-3}(x+20) = 2$
 b) $x(1 - \log_{21} 3) = \log_{21} 30 - \log_{21}(7^x + 1)$



Practice Problems

1. Simplify each of the following expressions.
- a) $\log_{10} 5 + \log_{10} 2$ e) $4^{\log_2 a}$ i) $\log_5(3x) + \log_5(15x^2) - 2\log_5 3$
 b) $\log_4 320 - \log_4 5$ f) $2^{\log_4 y}$ j) $2 \ln \sqrt{m} + 3 \ln \sqrt[3]{m}$
 c) $\log_2(40a) - \log_2 5a$ g) $2 \ln \sqrt{x^2 - 1} - \ln(x+1)$ k) $2 \log_3(2A^5) - \log_9(144A^8)$
 d) $\log_{10} 0.0002 + \log_{10} 5$ h) $\log_6 \sqrt{12} + \log_6 \sqrt{18}$ l) $\log_3(12b^2) - 2 \log_3(2b)$
- m) $\log \sqrt{52} + 3 \log 2 + \log 125 + \log \sqrt{325} - \log 13$
2. Which of the following is NOT equivalent to $\log_9 \left(\frac{36}{25}\right)$?
- A) $\frac{\ln \left(\frac{36}{25}\right)}{\ln 9}$ B) $\frac{\ln \left(\frac{6}{5}\right)}{2 \ln 3}$ C) $\frac{\ln 36 - \ln 25}{2 \ln 3}$ D) $\frac{\ln 36 - \ln 25}{\ln 9}$ E) $\frac{\ln 6 - \ln 5}{\ln 3}$

3. Let $x = \log_2 5$. Express each of the following in terms of x .

a) $\log_2 125$ c) $\log_2 1000$ e) $\log_2 \left(\frac{5}{2}\right)$ g) $\log_{10} 2$ i) $\log_{50} 80$ k) $\log_2 \left(\frac{16}{25}\right)$
 b) $\log_2 10$ d) $\log_2 80$ f) $\log_5 2$ h) $\log_5 10$ j) $\log_{100} 10$ l) $\log_5 \left(\frac{16}{25}\right)$

4. Let $p = \log_2 5$ and $q = \log_5 3$. Express each of the following in terms of p and q .

a) $\log_2 10$ c) $\log_5 10$ e) $\log_5 30$ g) $\log_2 24$
 b) $\log_5 45$ d) $\log_5 6$ f) $\log_2 3$ h) $\log_{24} 30$

5. Find the domain of each of the following expressions.

a) $\log_5 (-x^2 + 10x - 23)$ d) $\frac{1}{\log_2 (x-2) + \log_2 (x-4)}$ f) $\ln (x^2 - 6x) - \ln (4 - x^2)$
 b) $\log_2 (x^2 - 6x + 8)$
 c) $\frac{1}{\log_2 (x^2 - 6x + 8)}$ e) $\ln \left(\frac{x^2 - 6x}{4 - x^2}\right)$ g) $\frac{\ln (x^2 - 6x)}{\ln (4 - x^2)}$

6. Solve each of the following equations.

a) $\log_6 (8 - x) + \log_6 (x + 12) = 2$ f) $\log_{x-1} (x + 2) + \log_{x-1} (x - 2) = 2$
 b) $\log_4 (3m + 5) - \log_4 (m + 7) = \frac{1}{2}$ g) $\log_2 x + \log_2 (x - 4) = 5$
 c) $\log_2 (3x - 5) + \log_2 (x - 6) = 4$ h) $\log_4 (x - 1) + \log_4 (x + 3) = \frac{5}{2}$
 d) $\log_2 (2 - y) + \log_2 (10 - y) = 7$ i) $\log_6 x + \log_6 (2x + 1) = 2$
 e) $\log_x (12 - x) = 2$ j) $\log_2 (x - 5) + \log_2 (x + 11) = 9$



Answers

Sample Problems

1. a) 3 b) -1 c) $2 + 3 \log_{10} x$ d) -2 e) $1 + 8 \log_2 x$ f) 1 g) $\frac{3}{4}$ h) $\frac{14}{3}$ i) $\frac{1}{125}$ j) x^3 k) \sqrt{x} l) 2

2. E 3. see solutions

4. a) $x + 1$ b) $x + 2$ c) $2x + 1$ d) $3x + 1$ e) $3x + 2$ f) $\frac{1}{x}$ g) $\frac{3x + 1}{2x + 1}$ h) $x - 1$ i) $2 - 3x$ j) $\frac{3x + 1}{3x + 2}$

5. a) $\frac{2(a+1)}{3b+1}$ b) $\frac{3}{b} + \frac{a-1}{2} = \frac{ab-b+6}{2b}$ 6. a) 3 b) $\log_2 \left(\frac{\sqrt{10}}{2}\right)$ 7. see solutions

8. a) $\{x | x < -4 \text{ or } x > 4\}$ in interval notation: $(-\infty, -4) \cup (4, \infty)$ b) $\{x | x > 4\}$ in interval notation: $(4, \infty)$

c) $\{x | x > 3 \text{ but } x \neq 4\}$ in interval notation: $(3, \infty) \setminus \{4\}$ d) \mathbb{R} e) $\left\{x | x > \frac{1}{2} \text{ but } x \neq 41\right\}$ in interval notation: $\left(\frac{1}{2}, \infty\right) \setminus \{41\}$

f) $\{x | 0 < x < 2 \text{ but } x \neq 1\}$ in interval notation: $(0, 2) \setminus \{1\}$

9. a) -5, 7 b) 7 c) 35 d) 7 e) no solution f) 16 g) 16, 512 10.. a) $\frac{1}{\sqrt{8}}, 2$ b) $\log_7 5$ c) 5

Practice Problems

1. a) 1 b) 3 c) 3 d) -3 e) a^2 f) \sqrt{y} g) $x - 1$ h) $\frac{3}{2}$ i) $1 + 3 \log_5 x$ j) $2 \ln m$ k) $-1 + 6 \log_3 A$ l) 1 m) 4
2. B 3. a) $3x$ b) $x + 1$ c) $3x + 3$ d) $x + 4$ e) $x - 1$ f) $\frac{1}{x}$ g) $\frac{1}{x + 1}$ h) $\frac{x + 1}{x}$ i) $\frac{x + 4}{2x + 1}$ j) $\frac{1}{2}$ k) $4 - 2x$ l) $\frac{4 - 2x}{x} = \frac{4}{x} - 2$
4. a) $p + 1$ b) $2q + 1$ c) $\frac{1}{p} + 1$ d) $\frac{1}{p} + q$ e) $\frac{1}{p} + q + 1$ f) pq g) $pq + 3$ h) $\frac{\frac{1}{3} + q + 1}{\frac{3}{p} + q} = \frac{p + pq + 1}{pq + 3}$
5. a) $\{x \mid 5 - \sqrt{2} < x < 5 + \sqrt{2}\}$ - in interval notation: $(5 - \sqrt{2}, 5 + \sqrt{2})$
 b) $\{x \mid x < 2 \text{ or } x > 4\}$ - in interval notation: $(-\infty, 2) \cup (4, \infty)$
 c) $\{x < 2 \text{ and } x \neq 3 - \sqrt{2} \text{ or } x > 4 \text{ and } x \neq 3 + \sqrt{2}\}$ in interval notation: $(-\infty, 2) \cup (4, \infty) \setminus \{3 - \sqrt{2}, 3 + \sqrt{2}\}$
 d) $\{x > 4 \text{ and } x \neq 3 + \sqrt{2}\}$ - in interval notation: $(4, \infty) \setminus \{3 + \sqrt{2}\}$
 e) $\{x \mid -2 < x < 0 \text{ or } 2 < x < 6\}$ -in interval notation: $(-2, 0) \cup (2, 6)$
 f) $\{x \mid -2 < x < 0\}$ in interval notation: $(-2, 0)$ g) $\{-2 < x < 0 \text{ and } x \neq -\sqrt{3}\}$ in interval notation: $(-2, 0) \setminus \{-\sqrt{3}\}$
6. a) $-10, 6$ b) 9 c) 7 d) -6 e) 3 f) $\frac{5}{2}$ g) 8 h) 5 i) 4 j) 21

Sample Problems Solutions

1. Simplify each of the following expressions.

a) $\log_6 4 + \log_6 54 = \log_6 (4 \cdot 54) = \log_6 216 = 3$

b) $1 + 2 \log_2 3 - \log_2 36$

Solution: We re-write each expression as a single base 2 logarithm. We will use by the rule $n \log_a b = \log_a (b^n)$

$$1 = \log_2 2 \quad \text{and} \quad 2 \log_2 3 = \log_2 3^2 = \log_2 9$$

$$1 + 2 \log_2 3 - \log_2 36 = \log_2 2 + \log_2 9 - \log_2 36 = \log_2 \left(\frac{2 \cdot 9}{36} \right) = \log_2 \left(\frac{1}{2} \right) = -1$$

c) $2 \log_{10} (2x) + \log_{10} (25x)$

Solution: by the rule $n \log_a b = \log_a (b^n)$, we have $2 \log_{10} (2x) = \log_{10} [(2x)^2] = \log_{10} (4x^2)$

$$\begin{aligned} 2 \log_{10} (2x) + \log_{10} (25x) &= \log_{10} (4x^2) + \log_{10} (25x) = \log_{10} (4x^2 \cdot 25x) = \log_{10} (100x^3) \\ &= \log_{10} 100 + \log_{10} (x^3) = 2 + 3 \log_{10} x \end{aligned}$$

d) $\log 21 - \frac{1}{2} \log 28 - \log 15 - \log \sqrt{700}$

Solution: Note that $\log 21$ is the same as $\log_{10} 21$

$$\begin{aligned} E &= \log 21 - \frac{1}{2} \log 28 - \log 15 - \log \sqrt{700} = \log 21 - \log \sqrt{28} - \log 15 - \log \sqrt{700} \\ &= \log 21 - \left(\log \sqrt{28} + \log 15 + \log \sqrt{700} \right) = \log 21 - \left(\log \sqrt{28} \cdot 15 \cdot \sqrt{700} \right) \\ &= \log \frac{21}{\sqrt{28} \cdot 15 \cdot \sqrt{700}} = \log \frac{21}{2\sqrt{7} \cdot 15 \cdot 10\sqrt{7}} = \log \frac{3 \cdot 7}{2 \cdot 7 \cdot 15 \cdot 10} = \log \frac{3}{300} = \log \frac{1}{100} = -2 \end{aligned}$$

e) $2 \log_2 (2x^5) - \log_4 (144x^8) + \frac{1}{3} \log_2 (216x^6)$

Solution: We can combine the expressions only if they are simple logarithms of the same base. Recall the rule $n \log_a b = \log_a (b^n)$

$$2 \log_2 (2x^5) = \log_2 (2x^5)^2 = \log_2 (4x^{10})$$

We change the second expression to base 2.

$$\log_4 (144x^8) = \frac{\log_2 (144x^8)}{\log_2 4} = \frac{\log_2 (144x^8)}{2} = \frac{1}{2} \log_2 (144x^8)$$

and we use the rule $n \log_a b = \log_a (b^n)$ to get rid of the coefficient

$$\frac{1}{2} \log_2 (144x^8) = \log_2 \left[(144x^8)^{1/2} \right] = \log_2 \sqrt{144x^8} = \log_2 (12x^4)$$

We similarly get rid of $\frac{1}{3}$ in the third expression:

$$\frac{1}{3} \log_2 (216x^6) = \log_2 \left[(216x^6)^{1/3} \right] = \log_2 \sqrt[3]{216x^6} = \log_2 (6x^2)$$

We are now ready to simplify the expression:

$$\begin{aligned} E &= 2 \log_2 (2x^5) - \log_4 (144x^8) + \frac{1}{3} \log_2 (216x^6) \\ &= \log_2 (4x^{10}) - \log_2 (12x^4) + \log_2 (6x^2) \end{aligned}$$

And now we use $\log_a b - \log_a c = \log_a \left(\frac{b}{c}\right)$

$$= \log_2 \left(\frac{4x^{10}}{12x^4}\right) + \log_2 (6x^2) = \log_2 \left(\frac{x^6}{3}\right) + \log_2 (6x^2)$$

And now we use $\log_a b + \log_a c = \log_a (bc)$

$$= \log_2 \left(\frac{x^6}{3}\right) (6x^2) = \log_2 \frac{x^6 (6x^2)}{3} = \log_2 2x^8$$

Now we use $\log_a b + \log_a c = \log_a (bc)$ and $n \log_a b = \log_a b^n$ again.

$$\log_2 2x^8 = \log_2 2 + \log_2 x^8 = 1 + 8 \log_2 x$$

f) $\log_a \left(\left(3 - \frac{3a-2}{a+1} \right) \cdot \frac{a^2+a}{5} \right)$

Solution:

$$\begin{aligned} E &= \log_a \left(\left(3 - \frac{3a-2}{a+1} \right) \cdot \frac{a^2+a}{5} \right) = \log_a \left(\left(\frac{3(a+1)}{a+1} - \frac{3a-2}{a+1} \right) \cdot \frac{a^2+a}{5} \right) \\ &= \log_a \left(\frac{3(a+1) - (3a-2)}{a+1} \cdot \frac{a(a+1)}{5} \right) = \log_a \left(\frac{3a+3-3a+2}{1} \cdot \frac{a}{5} \right) = \log_a \left(5 \cdot \frac{a}{5} \right) = \log_a a = 1 \end{aligned}$$

g) $\log_9 \sqrt{27}$

Solution: We have seen problems like this in the previous logarithms lecture notes (logarithms 1) but the change base theorem makes solving it much easier. We simply switch to base 3.

$$\log_9 \sqrt{27} = \frac{\log_3 \sqrt{27}}{\log_3 9} = \frac{\frac{3}{2}}{2} = \frac{3}{4}$$

h) $\log_{\sqrt{m}} \sqrt[3]{m^7}$

Solution: We will switch to base m .

$$\log_{\sqrt{m}} \sqrt[3]{m^7} = \frac{\log_m \sqrt[3]{m^7}}{\log_m \sqrt{m}} = \frac{\log_m (m^{7/3})}{\log_m (m^{1/2})} = \frac{\frac{7}{3}}{\frac{1}{2}} = \frac{7}{3} \cdot \frac{2}{1} = \frac{14}{3}$$

i) $e^{-3 \ln 5}$

Solution: Recall that $a^{\log_a b} = b$. Thus $e^{\ln x} = x$.

$$e^{-3 \ln 5} = \left(e^{\ln 5} \right)^{-3} = 5^{-3} = \frac{1}{125}$$

j) $8^{\log_2 x}$

Solution: Recall that $a^{\log_a b} = b$. Thus $2^{\log_2 x} = x$

$$8^{\log_2 x} = (2^3)^{\log_2 x} = 2^{3 \log_2 x} = (2^{\log_2 x})^3 = x^3$$

This problem is about matching the base of the exponentiation with the base of the logarithm. There is another way of solving this problem now that we have the switch-base theorem. We can switch to base 8.

$$\log_2 x = \frac{\log_8 x}{\log_8 2} = \frac{\log_8 x}{\frac{1}{3}} = 3 \log_8 x \text{ and so } 8^{\log_2 x} = 8^{3 \log_8 x} = (8^{\log_8 x})^3 = x^3$$

k) $3^{\log_9 x} = (9^{1/2})^{\log_9 x} = (9)^{\frac{1}{2} \log_9 x} = (9^{\log_9 x})^{\frac{1}{2}} = x^{\frac{1}{2}} = \sqrt{x}$

Solution: We can either change the base of exponentiation

$$3^{\log_9 x} = (9^{1/2})^{\log_9 x} = (9)^{\frac{1}{2} \log_9 x} = (9^{\log_9 x})^{\frac{1}{2}} = x^{\frac{1}{2}} = \sqrt{x}$$

or change the base of the logarithm:

$$\log_9 x = \frac{\log_3 x}{\log_3 9} = \frac{\log_3 x}{2} = \frac{1}{2} \log_3 x$$

$$3^{\log_9 x} = 3^{(1/2) \log_3 x} = (3^{\log_3 x})^{1/2} = x^{\frac{1}{2}} = \sqrt{x}$$

l) $(\log_3 4)(\log_4 5)(\log_5 6)(\log_6 7)(\log_7 8)(\log_8 9)$

Solution: We use the change-base formula for logarithms to re-write the expression

$$\log_a b = \frac{\ln b}{\ln a}$$

$$\begin{aligned} E &= (\log_3 4)(\log_4 5)(\log_5 6)(\log_6 7)(\log_7 8)(\log_8 9) \\ &= \frac{\ln 4}{\ln 3} \cdot \frac{\ln 5}{\ln 4} \cdot \frac{\ln 6}{\ln 5} \cdot \frac{\ln 7}{\ln 6} \cdot \frac{\ln 8}{\ln 7} \cdot \frac{\ln 9}{\ln 8} = \frac{\ln 9}{\ln 3} = \frac{\ln 3^2}{\ln 3} = \frac{2 \ln 3}{\ln 3} = 2 \end{aligned}$$

2. Which of the following is NOT equivalent to $\log_8 \left(\frac{50}{3} \right)$?

A) $\frac{\ln \left(\frac{50}{3} \right)}{\ln 8}$ B) $\frac{\ln 50 - \ln 3}{\ln 8}$ C) $\frac{\ln 50 - \ln 3}{3 \ln 2}$ D) $\frac{2 \ln 5 + \ln 2 - \ln 3}{3 \ln 2}$ E) $\frac{2 \ln 5 - \ln 3}{3}$

Solution: Let us use the change base theorem to change to natural base.

$$\log_8 \left(\frac{50}{3} \right) = \frac{\ln \left(\frac{50}{3} \right)}{\ln 8} \text{ which is choice A}$$

Now we use the rule $\log_a b - \log_a c = \log_a \left(\frac{b}{c} \right)$ in the numerator.

$$\frac{\ln \left(\frac{50}{3} \right)}{\ln 8} = \frac{\ln 50 - \ln 3}{\ln 8} \text{ which is choice B}$$

Now we use the rule $n \log_a b = \log_a (b^n)$ in the denominator.

$$\frac{\ln 50 - \ln 3}{\ln 8} = \frac{\ln 50 - \ln 3}{\ln 2^3} = \frac{\ln 50 - \ln 3}{3 \ln 2} \quad \text{which is choice C}$$

Now we apply the rules $n \log_a b = \log_a (b^n)$ and $\log_a b + \log_a c = \log_a (bc)$ in the denominator to re-write $\ln 50$.

$$\begin{aligned} \ln 50 &= \ln (2 \cdot 5^2) = \ln 2 + \ln (5^2) = \ln 2 + 2 \ln 5 \\ \frac{\ln 50 - \ln 3}{3 \ln 2} &= \frac{2 \ln 5 + \ln 2 - \ln 3}{3 \ln 2} \quad \text{which is choice D} \end{aligned}$$

At this point, choice E is the only expression possibly not equivalent to $\log_8 \left(\frac{50}{3} \right)$. Indeed, choice E represents a serious algebraic error in simplifying our expression in D. Since

$$\frac{a + b - c}{3b} \neq \frac{a - c}{3}$$

E is NOT equivalent to the other expressions. One way to verify this is to enter these expressions into the calculator and see that the decimal approximations are all the same for $\log_8 \left(\frac{50}{3} \right)$ and choices A, B, C, and D, but different for E.

3. Prove that $\log_{(8/15)} \left(\frac{24}{25} \right) = \frac{3 \ln 2 + \ln 3 - 2 \ln 5}{3 \ln 2 - \ln 3 - \ln 5}$.

Solution: We first switch to natural logarithm.

$$\log_{(8/15)} \left(\frac{24}{25} \right) = \frac{\ln \left(\frac{24}{25} \right)}{\ln \left(\frac{8}{15} \right)} = \frac{\ln 24 - \ln 25}{\ln 8 - \ln 15} = \frac{\ln (2^3 \cdot 3) - \ln (5^2)}{\ln (2^3) - \ln (3 \cdot 5)} = \frac{\ln (2^3) + \ln 3 - 2 \ln 5}{3 \ln 2 - (\ln 3 + \ln 5)} = \frac{3 \ln 2 + \ln 3 - 2 \ln 5}{3 \ln 2 - \ln 3 - \ln 5}$$

4. Let $x = \log_3 2$. Express each of the following in terms of x .

a) $\log_3 6$

Solution: Recall that $\log_a (bc) = \log_a b + \log_a c$

$$\log_3 6 = \log_3 (3 \cdot 2) = \log_3 3 + \log_3 2 = 1 + x = \boxed{x + 1}$$

b) $\log_3 18$

Solution: Recall that $\log_a (bc) = \log_a b + \log_a c$

$$\log_3 18 = \log_3 (9 \cdot 2) = \log_3 9 + \log_3 2 = 2 + x = \boxed{x + 2}$$

c) $\log_3 12$

Solution: Recall that $\log_a (bc) = \log_a b + \log_a c$ and $\log_a (b^n) = n \log_a b$

$$\log_3 12 = \log_3 (3 \cdot 4) = \log_3 (3 \cdot 2^2) = \log_3 3 + \log_3 (2^2) = 1 + 2 \log_3 2 = 1 + 2x = \boxed{2x + 1}$$

d) $\log_3 24$

Solution: Recall that $\log_a (bc) = \log_a b + \log_a c$ and $\log_a (b^n) = n \log_a b$

$$\log_3 24 = \log_3 (3 \cdot 8) = \log_3 (3 \cdot 2^3) = \log_3 3 + \log_3 (2^3) = 1 + 3 \log_3 2 = 1 + 3x = \boxed{3x + 1}$$

e) $\log_3 72$ Solution: Recall that $\log_a (bc) = \log_a b + \log_a c$ and $\log_a (b^n) = n \log_a b$

$$\log_3 72 = \log_3 (9 \cdot 8) = \log_3 (3^2 \cdot 2^3) = \log_3 (3^2) + \log_3 (2^3) = 2 + 3 \log_3 2 = 2 + 3x = \boxed{3x + 2}$$

f) $\log_2 3$ Solution: We will change base to 3 using the change base formula. Recall that $\log_a b = \frac{\log_c b}{\log_c a}$.

$$\log_2 3 = \frac{\log_3 3}{\log_3 2} = \frac{1}{\log_3 2} = \boxed{\frac{1}{x}}$$

Note that this is a useful piece of information: if we swap the two numbers in a logarithm, we obtain the opposite of the original logarithm. In short, $\log_x y$ and $\log_y x$ are reciprocals.

$$\text{g) } \log_{12} 24 = \frac{\log_3 24}{\log_3 12} = \frac{3x + 1}{2x + 1}$$

Solution: We will change base to 3 using the change base formula. Recall that $\log_a b = \frac{\log_c b}{\log_c a}$. Also note that the expressions $\log_3 24$ and $\log_3 12$ were already worked out in previous problems.

$$\log_{12} 24 = \frac{\log_3 24}{\log_3 12} = \boxed{\frac{3x + 1}{2x + 1}}$$

h) $\log_3 \left(\frac{2}{3}\right)$ Solution: Recall that $\log_a \left(\frac{b}{c}\right) = \log_a b - \log_a c$

$$\log_3 \left(\frac{2}{3}\right) = \log_3 2 - \log_3 3 = \boxed{x - 1}$$

i) $\log_3 \left(\frac{9}{8}\right)$

$$\log_3 \left(\frac{9}{8}\right) = \log_3 9 - \log_3 8 = 2 - \log_3 (2^3) = 2 - 3 \log_3 2 = 2 - 3x = \boxed{-3x + 2}$$

Solution: Recall that $\log_a \left(\frac{b}{c}\right) = \log_a b - \log_a c$ and $\log_a (b^n) = n \log_a b$

$$\text{j) } \log_{72} 24 = \frac{\log_3 24}{\log_3 72} = \frac{3x + 1}{3x + 2}$$

Solution: We will change base to 3 using the change base formula. Recall that $\log_a b = \frac{\log_c b}{\log_c a}$. Also note that the expressions $\log_3 24$ and $\log_3 72$ were already worked out in previous problems.

$$\log_{72} 24 = \frac{\log_3 24}{\log_3 72} = \boxed{\frac{3x + 1}{3x + 2}}$$

5. a) Suppose that $\log_2 6 = a$ and $\log_8 5 = b$. Express $\log_{10} 144$ in terms of a and b .

Solution: We will only need to find $\log_2 3$ and $\log_2 5$ in terms of a and b .

$$a = \log_2 6 = \log_2 (2 \cdot 3) = \log_2 2 + \log_2 3 = 1 + \log_2 3$$

$$a = \log_2 3 + 1 \implies \log_2 3 = a - 1$$

$$b = \log_8 5 = \frac{\log_2 5}{\log_2 8} = \frac{\log_2 5}{3} \implies \log_2 5 = 3b$$

$$\begin{aligned} \log_{10} 144 &= \frac{\log_2 144}{\log_2 10} = \frac{\log_2 (16 \cdot 9)}{\log_2 (2 \cdot 5)} = \frac{\log_2 16 + \log_2 9}{\log_2 2 + \log_2 5} = \frac{4 + \log_2 3^2}{1 + \log_2 5} = \frac{4 + 2 \log_2 3}{1 + \log_2 5} \\ &= \frac{4 + 2(a - 1)}{1 + 3b} = \frac{4 + 2a - 2}{3b + 1} = \frac{2a + 2}{3b + 1} \end{aligned}$$

b) Let $a = \log_3 75$ and $b = \log_2 27$. Express $\log_3 10$ in terms of a and b .

Solution:

$$a = \log_3 75 = \log_3 (3 \cdot 25) = \log_3 3 + \log_3 25 = 1 + \log_3 5^2 = 1 + 2 \log_3 5$$

$$a = 1 + 2 \log_3 5$$

$$a - 1 = 2 \log_3 5 \implies \log_3 5 = \frac{a - 1}{2}$$

$$b = \log_2 27 = \frac{\log_3 27}{\log_3 2} = \frac{3}{\log_3 2} \implies b = \frac{3}{\log_3 2} \implies b \log_3 2 = 3 \implies \log_3 2 = \frac{3}{b}$$

$$\log_3 10 = \log_3 2 + \log_3 5 = \frac{3}{b} + \frac{a - 1}{2} = \frac{6}{2b} + \frac{(a - 1)b}{2b} = \frac{6 + ab - b}{2b}$$

6. a) Simplify $\frac{\log_3 90}{\log_{30} 3} - \frac{\log_3 270}{\log_{10} 3}$.

Solution: We first switch to base 3 and simplify the logarithms as much as possible.

$$\begin{aligned} \frac{\log_3 90}{\log_{30} 3} - \frac{\log_3 270}{\log_{10} 3} &= \frac{\log_3 90}{\frac{\log_3 3}{\log_3 30}} - \frac{\log_3 270}{\frac{\log_3 3}{\log_3 10}} = \frac{\log_3 90}{\frac{1}{\log_3 30}} - \frac{\log_3 270}{\frac{1}{\log_3 10}} \\ &= \frac{2 + \log_3 10}{\frac{1}{1 + \log_3 10}} - \frac{3 + \log_3 10}{\frac{1}{\log_3 10}} = (2 + \log_3 10)(1 + \log_3 10) - (3 + \log_3 10) \log_3 10 \\ &= 2 + 2 \log_3 10 + \log_3 10 + (\log_3 10)^2 - 3 \log_3 10 - (\log_3 10)^2 \\ &= 2 + 3 \log_3 10 - 3 \log_3 10 = 2 \end{aligned}$$

b) Write $\log_2 5 - \log_4 10$ as a single logarithm.

Solution: We first switch to base 2.

$$\begin{aligned}\log_2 5 - \log_4 10 &= \log_2 5 - \frac{\log_2 10}{\log_2 4} = \log_2 5 - \frac{\log_2 10}{2} = \log_2 5 - \frac{1}{2} \log_2 10 = \log_2 5 - \log_2 (10)^{1/2} \\ &= \log_2 5 - \log_2 \sqrt{10} = \log_2 \left(\frac{5}{\sqrt{10}} \right) = \log_2 \left(\frac{5\sqrt{10}}{10} \right) = \log_2 \left(\frac{\sqrt{10}}{2} \right)\end{aligned}$$

7. a) Prove that $\log_{(a^k)} (b^k) = \log_a b$.

Prove: We use the conversion formula for logarithms, to switch to base a .

$$\log_{(a^k)} (b^k) = \frac{\log_a (b^k)}{\log_a (a^k)} = \frac{k \log_a b}{k} = \log_a b$$

b) Prove that $\log_{a/b} \left(\frac{c}{d} \right) = \log_{b/a} \left(\frac{d}{c} \right)$.

Prove: We use the conversion formula for logarithms, to switch to the natural logarithm and then back. Also, we will use the following fact: $\frac{x-y}{z-w} = \frac{y-x}{w-z}$. This is true because

$$\frac{x-y}{z-w} = \frac{-1(-x+y)}{-1(-z+w)} = \frac{y-x}{w-z}$$

$$\log_{a/b} \left(\frac{c}{d} \right) = \frac{\ln \left(\frac{c}{d} \right)}{\ln \left(\frac{a}{b} \right)} = \frac{\ln c - \ln d}{\ln a - \ln b} = \frac{\ln d - \ln c}{\ln b - \ln a} = \frac{\ln \left(\frac{d}{c} \right)}{\ln \left(\frac{b}{a} \right)} = \log_{b/a} \left(\frac{d}{c} \right)$$

8. Find the domain of each of the following expressions.

a) $\log_3 (x^2 - 16)$

Solution: We need to solve the inequality $x^2 - 16 > 0$. (If you need to review these, see Quadratic Inequalities.) The solution is $\{x|x < -4 \text{ or } x > 4\}$ or in interval notation, $(-\infty, -4) \cup (4, \infty)$.

b) $\log_3 (x + 4) + \log_3 (x - 4)$

Solution: We need to solve the inequalities $x + 4 > 0$ and $x - 4 > 0$.

$$\begin{aligned}x + 4 &> 0 \text{ and } x - 4 > 0 \\ x &> -4 \text{ and } x > 4 \implies x > 4\end{aligned}$$

Thus the domain is $\{x|x > 4\}$ or in interval notation, $(4, \infty)$

$$c) \frac{1}{\ln(x-3)}$$

Solution: for the expression $\ln(x-3)$ to be defined, we need that $x-3 > 0$, thus $x > 3$. Now if x is greater than 3, $\ln(x-3)$ is defined but we still need to worry about division by zero. We have to rule out all values of x for which $\ln(x-3) = 0$. So we solve the equation

$$\begin{aligned} \ln(x-3) &= 0 & 1 &= x-3 \\ e^0 &= x-3 & 4 &= x \end{aligned}$$

Thus the domain is: $\{x|x > 3 \text{ but } x \neq 4\}$ or in interval notation, $(3, \infty) \setminus \{4\}$

$$d) \log_3(x^2+1)$$

Solution: for this logarithm to be defined, $x^2+1 > 0$ needs to be true. Since this inequality is true for all real numbers, this expression's domain is the set of all real numbers, \mathbb{R} .

$$e) \frac{1}{\log_3(2x-1)-4}$$

Solution: for $\log_3(2x-1)$ to be defined, $2x-1 > 0$ needs to be true. We solve this inequality and get that $x > \frac{1}{2}$. Even if the logarithm is defined, we still need to worry about division by zero. We have to rule out all values of x for which $\log_3(2x-1)-4 = 0$. We solve the equation

$$\begin{aligned} \log_3(2x-1)-4 &= 0 & 2x-1 &= 81 \\ \log_3(2x-1) &= 4 & 2x &= 82 \\ 2x-1 &= 3^4 & x &= 41 \end{aligned}$$

So the domain of this expression is $\left\{x|x > \frac{1}{2} \text{ but } x \neq 41\right\}$ or in interval notation, $\left(\frac{1}{2}, \infty\right) \setminus \{41\}$.

$$f) \frac{3}{\log_{10}(2x-x^2)}$$

Solution: for $\log_{10}(2x-x^2)$ to be defined, $2x-x^2 > 0$ needs to be true. We solve this inequality and get that $0 < x < 2$. Even if the logarithm is defined, we still need to worry about division by zero. We have to rule out all values of x for which $\log_{10}(2x-x^2) = 0$. We solve the equation

$$\begin{aligned} \log_{10}(2x-x^2) &= 0 & 0 &= x^2-2x+1 \\ 2x-x^2 &= 10^0 & 0 &= (x-1)^2 \\ 2x-x^2 &= 1 & x &= 1 \end{aligned}$$

So the domain of this expression is $\{x| 0 < x < 2 \text{ but } x \neq 1\}$ or in interval notation, $(0, 2) \setminus \{1\}$.

$$9. a) \log_2(x-3)(x+1) = 5$$

Solution: We re-write the logarithmic statement as an exponential statement and then solve for x .

$$\begin{aligned} (x-3)(x+1) &= 2^5 & x^2-2x-35 &= 0 \\ (x-3)(x+1) &= 32 & (x+5)(x-7) &= 0 \\ x^2-2x-3 &= 32 & x_1 &= -5 \quad x_2 = 7 \end{aligned}$$

We check: If $x = -5$, then

$$\text{LHS} = \log_2(-5-3)(-5+1) = \log_2(-8)(-4) = \log_2 32 = 5 = \text{RHS}$$

and if $x = 7$, then

$$\text{LHS} = \log_2(7-3)(7+1) = \log_2(4)(8) = \log_2 32 = 5 = \text{RHS}$$

$$\text{b) } \log_2(x-3) + \log_2(x+1) = 5$$

Solution: $\log_2(x-3) + \log_2(x+1) = \log_2(x-3)(x+1)$ and so this equation appears to be identical to the previous one. But it is not. Let's check: If $x = -5$, then

$$\text{LHS} = \log_2(-5-3) + \log_2(-5+1) = \log_2(-8) + \log_2(-4) = \text{undefined}$$

and if $x = 7$, then

$$\text{LHS} = \log_2(7-3) + \log_2(7+1) = \log_2 4 + \log_2 8 = 2 + 3 = 5 = \text{RHS}$$

and so this equation has only one solution, $x = 7$. So, it is very important to check.

$$\text{c) } \log_2(x+29) - \log_2(x-3) = 1$$

Solution:

$$\begin{aligned} \log_2(x+29) - \log_2(x-3) &= 1 & x+29 &= 2(x-3) \\ \log_2 \frac{x+29}{x-3} &= 1 & x+29 &= 2x-6 \\ \frac{x+29}{x-3} &= 2^1 & 35 &= x \end{aligned}$$

We check: if $x = 35$, then

$$\text{LHS} = \log_2(35+29) - \log_2(35-3) = \log_2 64 - \log_2 32 = 6 - 5 = 1 = \text{RHS}$$

$$\text{d) } \log_6 2 + \log_6(2x-5) + \log_6(x-5) = 2$$

Solution:

$$\begin{aligned} \log_6 2 + \log_6(2x-5) + \log_6(x-5) &= 2 \\ \log_6 2(2x-5)(x-5) &= 2 && \text{re-write as exponential} \\ 6^2 &= 2(2x-5)(x-5) \\ 36 &= 2(2x-5)(x-5) && \text{divide by 2} \\ 18 &= (2x-5)(x-5) \\ 18 &= 2x^2 - 15x + 25 \\ 0 &= 2x^2 - 15x + 7 && \text{factor} \\ 0 &= (2x-1)(x-7) \implies x_1 = \frac{1}{2} \text{ and } x_2 = 7 \end{aligned}$$

We check: if $x = \frac{1}{2}$, then

$$\text{LHS} = \log_6 2 + \log_6 \left(2 \left(\frac{1}{2} \right) - 5 \right) + \log_6 \left(\frac{1}{2} - 5 \right) = \log_6 2 + \log_6(-4) + \log_6(-4.5)$$

The left-hand side is undefined because the logarithm of negative numbers is NOT defined. Thus $x = \frac{1}{2}$ is NOT a solution. If $x = 7$, then

$$\text{LHS} = \log_6 2 + \log_6(2 \cdot 7 - 5) + \log_6(7 - 5) = \log_6 2 + \log_6 9 + \log_6 2 = \log_6(2 \cdot 9 \cdot 2) = \log_6 36 = 2 = \text{RHS}$$

Thus $x = 7$ is the only solution.

$$e) \log_2(x-3) - \log_2(x+1) = 1$$

Solution:

$$\begin{aligned} \log_2(x-3) - \log_2(x+1) &= 1 & x-3 &= 2(x+1) \\ \log_2 \frac{x-3}{x+1} &= 1 & x-3 &= 2x+2 \\ \frac{x-3}{x+1} &= 2 & -5 &= x \end{aligned}$$

We check: if $x = -5$, then

$$\text{LHS} = \log_2(-5-3) - \log_2(-5+1) = \log_2(-8) - \log_2(-4) = \text{undefined}$$

The only number, -5 that could work with this equation, doesn't and so this equation has no solution.

$$f) \left[64^{\frac{2}{3}} \cdot 3^{-\log_{27} 8} \right]^{\frac{1}{3}} + \log_2 x^3 = 14$$

Solution:

$$\begin{aligned} 64^{\frac{2}{3}} &= \left(\sqrt[3]{64} \right)^2 = 4^2 = 16 \\ \log_{27} 8 &= \frac{\ln 8}{\ln 27} = \frac{\ln 2^3}{\ln 3^3} = \frac{3 \ln 2}{3 \ln 3} = \frac{\ln 2}{\ln 3} = \log_3 2 \\ 3^{-\log_{27} 8} &= 3^{-\log_3 2} = \frac{1}{3^{\log_3 2}} = \frac{1}{3^{\log_3 2}} = \frac{1}{2} \end{aligned}$$

$$\begin{aligned} \left(16 \cdot \frac{1}{2} \right)^{\frac{1}{3}} + \log_2 x^3 &= 14 & 3 \log_2 x &= 12 \\ 8^{\frac{1}{3}} + 3 \log_2 x &= 14 & \log_2 x &= 4 \\ 2 + 3 \log_2 x &= 14 & x &= 16 \end{aligned}$$

$$g) \log_{64} x + \log_x 64 = \frac{13}{6}$$

Solution: The trick is to realize that $\log_{64} x$ and $\log_x 64$ are reciprocals since

$$\log_{64} x = \frac{\log_2 x}{\log_2 64} = \frac{\log_2 x}{6} \quad \text{and} \quad \log_x 64 = \frac{\log_2 64}{\log_2 x} = \frac{6}{\log_2 x}$$

If we denote $a = \log_{64} x$, then we have $a + \frac{1}{a} = \frac{13}{6}$ where $a \neq 0$. We solve this equation using the quadratic formula (completing the square would also work).

$$\begin{aligned} a + \frac{1}{a} &= \frac{13}{6} \\ a^2 - \frac{13}{6}a + 1 &= 0 \\ 6a^2 - 13a + 6 &= 0 \end{aligned}$$

$$a_{1,2} = \frac{13 \pm \sqrt{(-13)^2 - 4 \cdot 6 \cdot 6}}{2 \cdot 6} = \frac{13 \pm \sqrt{169 - 144}}{12} = \frac{13 \pm \sqrt{25}}{12} = \frac{13 \pm 5}{12} = \frac{2}{3} \quad \text{or} \quad \frac{3}{2}$$

We check:

$$\text{If } x = \frac{2}{3}, \text{ LHS} = \frac{2}{3} + \frac{1}{\left(\frac{2}{3}\right)} = \frac{2}{3} + \frac{3}{2} = \frac{4}{6} + \frac{9}{6} = \frac{13}{6} = \text{RHS}$$

$$\text{If } x = \frac{3}{2}, \text{ LHS} = \frac{3}{2} + \frac{1}{\left(\frac{3}{2}\right)} = \frac{3}{2} + \frac{2}{3} = \frac{9}{6} + \frac{4}{6} = \frac{13}{6} = \text{RHS}$$

So now we have $a = \frac{2}{3}$ or $\frac{3}{2}$. Since $a = \log_{64} x$, we have: if $a = \frac{2}{3}$

$$\log_{64} x = \frac{2}{3} \implies 64^{2/3} = x \implies x = 64^{2/3} = \left(\sqrt[3]{64}\right)^2 = 4^2 = 16$$

Or, if $a = \frac{3}{2}$, then

$$\log_{64} x = \frac{3}{2} \implies 64^{3/2} = x \implies x = 64^{3/2} = \left(\sqrt{64}\right)^3 = 8^3 = 512$$

10. (Enrichment) Solve each of the following equations.

a) $\log_{2x} 16 + \log_{4x} 8 = \log_x 8$

Solution:

$$\begin{aligned} \log_{2x} 16 + \log_{4x} 8 &= \log_x 8 && \text{switch to base 2} \\ \frac{\log_2 16}{\log_2 2x} + \frac{\log_2 8}{\log_2 4x} &= \frac{\log_2 8}{\log_2 x} \\ \frac{4}{1 + \log_2 x} + \frac{3}{2 + \log_2 x} &= \frac{3}{\log_2 x} && \text{Let } a \text{ denote } \log_2 x \\ \frac{4}{a+1} + \frac{3}{a+2} &= \frac{3}{a} && \text{multiply by } a(a+1)(a+2) \\ 4a(a+2) + 3a(a+1) &= 3(a+1)(a+2) \\ 4a^2 + 8a + 3a^2 + 3a &= 3(a^2 + 3a + 2) \\ 7a^2 + 11a &= 3a^2 + 9a + 6 \\ 4a^2 + 2a - 6 &= 0 \\ 2a^2 + a - 3 &= 0 \\ (2a+3)(a-1) &= 0 \\ a_1 = -\frac{3}{2} \quad a_2 = 1 \end{aligned}$$

So $x_1 = 2^{-3/2} = \frac{1}{\sqrt{8}}$ and $x_2 = 2$

$$\text{b) } x(1 - \log_{21} 3) = \log_{21} 30 - \log_{21} (7^x + 1)$$

Solution:

$$\begin{aligned} x(1 - \log_{21} 3) &= \log_{21} 30 - \log_{21} (7^x + 1) \\ x(\log_{21} 21 - \log_{21} 3) &= \log_{21} 30 - \log_{21} (7^x + 1) \\ x \log_{21} \frac{21}{3} &= \log_{21} 30 - \log_{21} (7^x + 1) \\ x \log_{21} 7 &= \log_{21} 30 - \log_{21} (7^x + 1) \\ \log_{21} (7^x) &= \log_{21} \left(\frac{30}{7^x + 1} \right) \end{aligned}$$

Since $f(x) = \log_{21} x$ is a one-to-one function, we can conclude that

$$\begin{aligned} 7^x &= \frac{30}{7^x + 1} \\ 7^x (7^x + 1) &= 30 \end{aligned}$$

Let $a = 7^x$

$$\begin{aligned} a(a + 1) &= 30 \\ a^2 + a - 30 &= 0 \\ (a + 6)(a - 5) &= 0 \\ a_1 = -6 \quad \text{and} \quad a_2 = 5 \end{aligned}$$

If $a = -6$, then we have $7^x = -6$. This equation has no solution. If $a = 5$, then we have $7^x = 5$ and so $x = \log_7 5$.

$$\text{c) } \log_x (x - 3) \cdot \log_{x-3} (x + 20) = 2$$

Solution: We will first change the base of the second logarithm to x .

$$\begin{aligned} \log_x (x - 3) \cdot \log_{x-3} (x + 20) &= 2 \\ \log_x (x - 3) \cdot \frac{\log_x (x + 20)}{\log_x (x - 3)} &= 2 \quad \text{cancel out } \log_x (x - 3) \\ \log_x (x + 20) &= 2 \\ x^2 &= x + 20 \\ x^2 - x - 20 &= 0 \\ (x - 5)(x + 4) &= 0 \\ x_1 = 5 \quad x_2 = -4 \end{aligned}$$

Since x is the base of a logarithm, and also $x - 3$ is the argument of a logarithm, $x = -4$ clearly does not work. We check the other solution and find that it does work.

$$\text{LHS} = \log_5 (5 - 3) \cdot \log_{5-3} (5 + 20) = \log_5 2 \cdot \log_2 25 = \log_5 2 \cdot (2 \log_2 5) = 2 (\log_5 2 \cdot \log_2 5) = 2 \cdot 1 = 2$$

So $x = 5$.