A **logarithmic equation** is an equation in which the unknown appears inside a logarithm, such as $\log_2(3x-1)=5$, or even in the base of a logarithm, such as $\log_x 81=4$. The basic ones can be solved with nothing more than the definition of the logarithm.

Definition. Suppose that $b>0$, $b\neq 1$, and $a>0$. The **logarithm** of a with base b, $\log_b a$, is the exponent we need to raise b to in order to get a:

$$\log_b a = c \qquad \text{means exactly the same as} \qquad b^c = a.$$

The logarithm with base 10 is called the **common logarithm**, written $\log a$, and the logarithm with base e is called the **natural logarithm**, written $\ln a$.

Notice the conditions in the definition. The **base** of a logarithm must be positive and different from 1, and we can only take the logarithm of a **positive number**. These conditions will be important when we check our solutions.

The definition gives us the method: we walk away from the logarithmic statement and write the *equivalent exponential statement* instead. After that, the logarithm is gone and we solve an equation we already know how to solve.

Type 1: One logarithm equals a number

Example 1. Solve the equation $\log_2(3x-1)=5$.

Solution: The statement says that 5 is the exponent we need to raise 2 to in order to get $3x-1$.

$$\begin{aligned}
\log_2(3x-1) &= 5 && \text{write the equivalent exponential statement}\\
3x-1 &= 2^5 &&\\
3x-1 &= 32 && \text{add 1}\\
3x &= 33 && \text{divide by 3}\\
x &= 11 &&
\end{aligned}$$

Check: $\log_2(3\cdot 11-1)=\log_2 32=5$, because $2^5=32$. The solution is $x=11$.

Example 2. Solve the equation $\log_3(2x+7)=-2$.

Solution: A negative exponent gives a fraction: $3^{-2}=\dfrac{1}{3^2}=\dfrac{1}{9}$.

$$\begin{aligned}
\log_3(2x+7) &= -2 && \text{write the equivalent exponential statement}\\
2x+7 &= 3^{-2}=\frac{1}{9} && \text{subtract } 7=\frac{63}{9}\\
2x &= -\frac{62}{9} && \text{divide by 2}\\
x &= -\frac{31}{9} &&
\end{aligned}$$

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Check: the solution is negative, but that is not a problem. What must be positive is the argument of the logarithm: $2\left(-\frac{31}{9}\right)+7=-\frac{62}{9}+\frac{63}{9}=\frac{1}{9}$, which is positive. The solution is $x=-\frac{31}{9}$.

Example 3. Solve the equation $\frac{3}{4}\log_2(5x+1)-1=5$.

Solution: Before we can write the equivalent exponential statement, we must isolate the logarithm: the 1 and the factor $\frac{3}{4}$ are not part of it.

$$\begin{aligned}
\frac{3}{4}\log_2(5x+1)-1 &= 5 && \text{add } 1\\
\frac{3}{4}\log_2(5x+1) &= 6 && \text{multiply by } \frac{4}{3}\\
\log_2(5x+1) &= 8 && \text{write the equivalent exponential statement}\\
5x+1 &= 2^8=256 && \text{subtract } 1\\
5x &= 255 && \text{divide by } 5\\
x &= 51
\end{aligned}$$

Check: $\frac{3}{4}\log_2(5\cdot 51+1)-1=\frac{3}{4}\log_2 256-1=\frac{3}{4}\cdot 8-1=5$. The solution is $x=51$.

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Example 4. Solve each of the following equations. a) $\frac{1}{2}\log_2(x+3)-1=2$ b) $\frac{\log_2(x+3)-1}{2}=2$

Solution: The two equations look almost the same, but the order of operations is different, and so is the solution. To isolate the logarithm, we undo the operations in the *reverse* order.

a) Here we take the logarithm, then multiply by $\frac{1}{2}$, and then subtract 1. So first we undo the subtraction, and then the multiplication.

$$\begin{aligned}
\frac{1}{2}\log_2(x+3)-1 &= 2 && \text{add 1}\\
\frac{1}{2}\log_2(x+3) &= 3 && \text{multiply by 2}\\
\log_2(x+3) &= 6 && \text{write the equivalent exponential statement}\\
x+3 &= 2^6=64 && \text{subtract 3}\\
x &= 61
\end{aligned}$$

b) Here the fraction bar groups the numerator: we take the logarithm, then subtract 1, and then divide by 2. So first we undo the division, and then the subtraction.

$$\begin{aligned}
\frac{\log_2(x+3)-1}{2} &= 2 && \text{multiply by 2}\\
\log_2(x+3)-1 &= 4 && \text{add 1}\\
\log_2(x+3) &= 5 && \text{write the equivalent exponential statement}\\
x+3 &= 2^5=32 && \text{subtract 3}\\
x &= 29
\end{aligned}$$

Check: a) $\frac{1}{2}\log_2 64-1=\frac{1}{2}\cdot 6-1=2$ b) $\frac{\log_2 32-1}{2}=\frac{5-1}{2}=2$

Example 5. Solve each of the following equations. a) $\log(2x-5)=2$ b) $\ln(x+3)=1$

Solution: a) When no base is written, the base is 10.

$$\begin{aligned}
\log(2x-5) &= 2 && \text{write the equivalent exponential statement}\\
2x-5 &= 10^2=100 && \text{add 5}\\
2x &= 105 && \text{divide by 2}\\
x &= \frac{105}{2}
\end{aligned}$$

b) The natural logarithm has base e.

$$\begin{aligned}
\ln(x+3) &= 1 && \text{write the equivalent exponential statement}\\
x+3 &= e^1=e && \text{subtract 3}\\
x &= e-3
\end{aligned}$$

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This is the exact value of the solution. (With a calculator, $e - 3 \approx -0.28$, and the argument $x + 3 = e$ is positive.)

Remark: An exponential equation such as $2^x = -4$ has no solution, because a power of a positive base is always positive. A logarithm, on the other hand, can be any number: positive, negative, or zero. So an equation like $\log_b(\text{something}) = c$ never fails because of the number c. What we must watch is the argument and the base.

Type 2: The argument is a quadratic expression

Example 6. Solve the equation $\log_2\left(x^2 - 2x\right) = 3$.

Solution: The method is the same, but after removing the logarithm we get a quadratic equation.

$$\begin{aligned} \log_2\left(x^2 - 2x\right) &= 3 && \text{write the equivalent exponential statement} \\ x^2 - 2x &= 2^3 = 8 && \text{subtract } 8 \\ x^2 - 2x - 8 &= 0 && \text{factor} \\ (x-4)(x+2) &= 0 \end{aligned}$$

So $x = 4$ or $x = -2$. Check: for $x = 4$, $x^2 - 2x = 16 - 8 = 8$, and for $x = -2$, $x^2 - 2x = 4 + 4 = 8$. In both cases the argument is 8, which is positive, and $\log_2 8 = 3$. Both numbers are solutions: $x = 4$ and $x = -2$.

This is not a coincidence: any solution of $x^2 - 2x = 8$ makes the argument equal to 8. When the equation has only one logarithm, the argument is automatically equal to b^c, which is positive.

Type 3: The unknown is in the base

Example 7. Solve the equation $\log_x 81 = 4$.

Solution: The statement says that 4 is the exponent we need to raise x to in order to get 81.

$$\log_x 81 = 4 \qquad \text{means} \qquad x^4 = 81$$

Both 3 and -3 satisfy $x^4 = 81$, because $3^4 = (-3)^4 = 81$. But the base of a logarithm must be positive and different from 1, so $x = -3$ is not a solution. The solution is $x = 3$.

Example 8. Solve each of the following equations. a) $\log_x 3 = \dfrac{1}{2}$ b) $\log_x 1 = 2$

Solution: a) $\log_x 3 = \dfrac{1}{2}$ means $x^{\frac{1}{2}} = 3$, that is, $\sqrt{x} = 3$. We square both sides: $x = 9$. This is positive and different from 1, so it is a good base. Indeed, $\log_9 3 = \dfrac{1}{2}$, because $9^{\frac{1}{2}} = \sqrt{9} = 3$.

b) $\log_x 1 = 2$ means $x^2 = 1$, so $x = 1$ or $x = -1$. But a base must be positive and different from 1, so neither number is a good base. This equation has **no solution**.

Summary. To solve a basic logarithmic equation,

1. isolate the logarithm.
2. Write the equivalent exponential statement: $\log_b a = c$ means $b^c = a$.
3. Solve the resulting equation.
4. Check: the argument of the logarithm must be positive, and the base must be positive and different from 1.



Practice Problems

Solve each of the following equations. Give the exact value of the solution.

1. $\log_3 x = 4$
2. $\log_2 x = -3$
3. $\log_5 x = 0$
4. $\log x = 3$
5. $\ln x = 2$
6. $\log_4 x = \dfrac{1}{2}$
7. $\log_2 (x+5) = 4$
8. $\log_3 (2x-1) = 2$
9. $\log_5 (3x+1) = 2$
10. $\log_2 (5x-3) = -1$
11. $\log (4x+2) = 1$
12. $\ln (2x-1) = 0$
13. $\ln (3x+1) = 2$
14. $2\log_3 (x-4) = 6$
15. $3\log_2 (x+1) - 4 = 5$
16. $\dfrac{1}{2}\log_5 (2x+3) = 1$
17. $\dfrac{\log_4 (3x-2)+1}{3} = 1$
18. $\dfrac{1}{3}\log_2 (x-1) + 2 = 3$
19. $\dfrac{\log_2 (x-1)+2}{3} = 3$
20. $\log_2 \left(x^2-1\right) = 3$
21. $\log_3 \left(x^2+2x\right) = 1$
22. $\log \left(x^2-15x\right) = 2$
23. $\log_x 25 = 2$
24. $\log_x 8 = 3$
25. $\log_x 16 = 4$
26. $\log_x \dfrac{1}{9} = -2$
27. $\log_x 5 = \dfrac{1}{2}$
28. $\log_x 4 = -\dfrac{1}{2}$
29. $\log_x 1 = 3$
30. The acidity of a solution is measured by its pH: $\text{pH} = -\log [\text{H}^+]$, where $[\text{H}^+]$ is the concentration of hydrogen ions (in moles per liter). Lemon juice has a pH of about 2.3. Find the exact value of $[\text{H}^+]$ in lemon juice.



Enrichment

Solve each of the following equations.

31. $\log_2 (\log_3 x) = 1$
32. $\log_3 (\log_2 x) = 0$
33. $(\log_2 x)^2 - \log_2 x - 6 = 0$
34. $(\ln x)^2 = 4 \ln x$
35. $\log_x (x+6) = 2$

Hint for problems 33 and 34: let $u = \log_2 x$ or $u = \ln x$.

36. A puzzle. Write any positive integer n using the number 2 exactly three times, and any mathematical operations you like. (The square root sign does not count as a 2.) Hint: first compute $\log_2 \log_2 \sqrt{\sqrt{2}}$.

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Answers

1. $x = 3^4 = 81$
2. $x = 2^{-3} = \dfrac{1}{8}$
3. $x = 5^0 = 1$
4. $x = 10^3 = 1000$
5. $x = e^2$
6. $x = 4^{\frac{1}{2}} = 2$
7. $x + 5 = 2^4 = 16,\ x = 11$
8. $2x - 1 = 3^2 = 9,\ x = 5$
9. $3x + 1 = 5^2 = 25,\ x = 8$
10. $5x - 3 = 2^{-1} = \dfrac{1}{2},\ x = \dfrac{7}{10}$
11. $4x + 2 = 10,\ x = 2$
12. $2x - 1 = e^0 = 1,\ x = 1$
13. $3x + 1 = e^2,\ x = \dfrac{e^2 - 1}{3}$
14. $\log_3(x - 4) = 3,\ x - 4 = 27,\ x = 31$
15. $\log_2(x + 1) = 3,\ x + 1 = 8,\ x = 7$
16. $\log_5(2x + 3) = 2,\ 2x + 3 = 25,\ x = 11$
17. $\log_4(3x - 2) = 2,\ 3x - 2 = 16,\ x = 6$
18. $\log_2(x - 1) = 3,\ x - 1 = 8,\ x = 9$
19. $\log_2(x - 1) = 7,\ x - 1 = 128,\ x = 129$
20. $x^2 - 1 = 8,\ x = 3$ or $x = -3$ (both are solutions)
21. $x^2 + 2x - 3 = 0,\ x = 1$ or $x = -3$ (both are solutions)
22. $x^2 - 15x - 100 = 0,\ x = 20$ or $x = -5$ (both are solutions)
23. $x^2 = 25,\ x = 5$ ($x = -5$ is not a good base)
24. $x^3 = 8,\ x = 2$
25. $x^4 = 16,\ x = 2$ ($x = -2$ is not a good base)
26. $x^{-2} = \dfrac{1}{9},\ x^2 = 9,\ x = 3$ ($x = -3$ is not a good base)
27. $\sqrt{x} = 5,\ x = 25$
28. $x^{-\frac{1}{2}} = 4,\ \sqrt{x} = \dfrac{1}{4},\ x = \dfrac{1}{16}$
29. $x^3 = 1,\ x = 1,$ but 1 is not a good base: no solution
30. $\log[\mathrm{H}^+] = -2.3,\ [\mathrm{H}^+] = 10^{-2.3}$ moles per liter (about 0.005)
31. $\log_3 x = 2,\ x = 9$
32. $\log_2 x = 1,\ x = 2$
33. $(u - 3)(u + 2) = 0,\ \log_2 x = 3$ or $\log_2 x = -2,$ $x = 8$ or $x = \dfrac{1}{4}$
34. $u(u - 4) = 0,\ \ln x = 0$ or $\ln x = 4,\ x = 1$ or $x = e^4$
35. $x^2 = x + 6,\ (x - 3)(x + 2) = 0,\ x = 3$ ($x = -2$ is not a good base)
36. $\sqrt{\sqrt{2}} = 2^{\frac{1}{4}},$ so $\log_2 \log_2 \sqrt{\sqrt{2}} = \log_2 \dfrac{1}{4} = -2.$ In general, n square roots give $2^{\frac{1}{2^n}},$ and $\log_2 2^{\frac{1}{2^n}} = \dfrac{1}{2^n} = 2^{-n},$ so $n = -\log_2 \log_2 \underbrace{\sqrt{\sqrt{\cdots\sqrt{2}}}}_{n \text{ square roots}}$

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