

Why Some Functions Can Be Reversed

Recall that a **function** f assigns to each input x in its domain exactly one output $f(x)$. Nothing in that definition prevents two *different* inputs from sharing the *same* output. For instance, if $f(x) = x^2$, then $f(-3) = 9$ and $f(3) = 9$: the single output 9 comes from two inputs.

A **one-to-one** function is one in which this never happens — each output comes from only one input. Such functions are exactly the ones that can be “run backward” unambiguously.

Definition: A function f is **one-to-one** (or **injective**) if no two distinct inputs produce the same output; that is, for all a and b in the domain of f ,

$$a \neq b \implies f(a) \neq f(b).$$

Equivalently, taking the contrapositive,

$$f(a) = f(b) \implies a = b.$$

The second form is the one used in computations: assume the outputs are equal and show the inputs must have been equal.

Example 1. Show that $f(x) = 5x - 4$ is one-to-one.

Solution. Suppose $f(a) = f(b)$. Then

$$5a - 4 = 5b - 4 \implies 5a = 5b \implies a = b.$$

Since $f(a) = f(b)$ forces $a = b$, the function is one-to-one.

Example 2. Show that $g(x) = x^2 - 6x$ is not one-to-one.

Solution. It is enough to find two different inputs with the same output. Since $g(0) = 0$ and $g(6) = 36 - 36 = 0$, we have $g(0) = g(6)$ but $0 \neq 6$. Hence g is not one-to-one.

Example 3. Show that $h(x) = \frac{2x+1}{x-3}$ (with $x \neq 3$) is one-to-one.

Solution. Assume $h(a) = h(b)$:

$$\frac{2a+1}{a-3} = \frac{2b+1}{b-3}.$$

Cross-multiplying,

$$(2a+1)(b-3) = (2b+1)(a-3).$$

Expanding both sides:

$$2ab - 6a + b - 3 = 2ab - 6b + a - 3.$$

Cancel $2ab$ and -3 from each side:

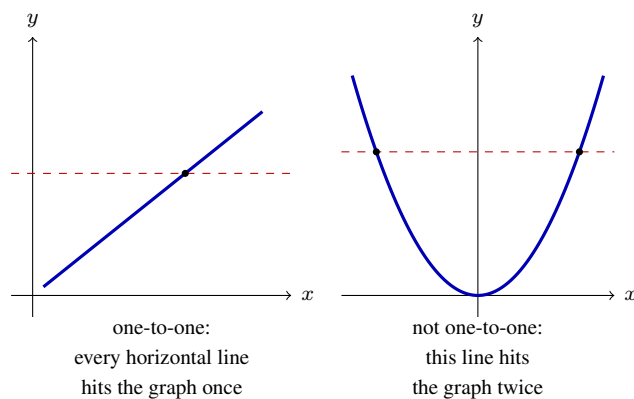
$$-6a + b = -6b + a \implies 7b = 7a \implies a = b.$$

Therefore h is one-to-one.

The Horizontal Line Test

The definition has a clean graphical meaning. If $f(a) = f(b) = c$ with $a \neq b$, then the points (a, c) and (b, c) both lie on the graph of f and both lie on the horizontal line $y = c$. So a function *fails* to be one-to-one exactly when some horizontal line meets its graph more than once.

Theorem: The Horizontal Line Test A function f is one-to-one if and only if no horizontal line intersects the graph of f at more than one point.



Example 4. The graph of $y = x^3$ passes the Horizontal Line Test — each horizontal line crosses it exactly once — so $f(x) = x^3$ is one-to-one. The graph of $y = |x|$ fails the test (the line $y = 2$ meets it at $(-2, 2)$ and $(2, 2)$), so $f(x) = |x|$ is not one-to-one.

Increasing and Decreasing Functions

There is a large, easily recognized supply of one-to-one functions.

Theorem: If f is strictly increasing on its entire domain, or decreasing on its entire domain, then f is one-to-one.

Proof: Suppose f is increasing and let $a \neq b$. One of them is smaller; say $a < b$. Because f is increasing, $a < b$ gives $f(a) < f(b)$, so in particular $f(a) \neq f(b)$. The decreasing case is identical with the inequality reversed.

Example 5. Each of the following is one-to-one because it is increasing or decreasing on all of its domain:

$$f(x) = 2x + 7, \quad f(x) = x^3, \quad f(x) = \sqrt{x}, \quad f(x) = e^x, \quad f(x) = \left(\frac{1}{2}\right)^x.$$

Remark: The converse of the theorem is false. The function

$$f(x) = \begin{cases} x, & x < 0, \\ x + 1, & x \geq 0, \end{cases}$$

is one-to-one (no output is repeated) yet it is not increasing on all of $\mathbb{R}$, since it jumps at $x = 0$.

Restricting the Domain

A function that is not one-to-one can often be *made* one-to-one by discarding part of its domain. This is how inverses are obtained for functions like x^2 .

Example 6. The function $f(x) = x^2$ is not one-to-one on $(-\infty, \infty)$. Restricted to the domain $[0, \infty)$, it is increasing, hence one-to-one. Restricted instead to $(-\infty, 0]$, it is decreasing, and again one-to-one. Each restriction keeps every output value $y \geq 0$ but supplies only one input for it.

An Algebraic Test: Summary

Deciding whether f is one-to-one.

1. **Graph available?** Apply the Horizontal Line Test.
2. **Recognizable shape?** If f is increasing on its whole domain, or decreasing on its whole domain, it is one-to-one.
3. **Algebraic proof that f is one-to-one:** assume $f(a) = f(b)$ and use valid algebra to deduce $a = b$.
4. **Algebraic proof that f is *not* one-to-one:** produce two specific inputs $a \neq b$ with $f(a) = f(b)$. (For a quadratic, values equally spaced on either side of the vertex work.)

Example 7. Decide whether each function is one-to-one.

1. $f(x) = 3 - \frac{1}{2}x$
2. $f(x) = x^4$
3. $f(x) = \sqrt{x-1}$

Solution.

1. Linear with nonzero slope, so decreasing on $\mathbb{R}$; one-to-one.
2. $f(-1) = 1 = f(1)$ with $-1 \neq 1$; *not* one-to-one.
3. Increasing on its domain $[1, \infty)$ (a rightward shift of $\sqrt{x}$); one-to-one. Algebraically: $\sqrt{a-1} = \sqrt{b-1}$ implies $a-1 = b-1$, so $a = b$.

Looking Ahead

If f is one-to-one, then reversing every ordered pair (x, y) on its graph again produces a function — the **inverse function** f^{-1} — because each y came from exactly one x . If f is not one-to-one, reversing the pairs produces a relation that fails the Vertical Line Test, so no inverse function exists without restricting the domain.



Practice Problems

Concept check.

1. State the definition of a one-to-one function in the form “ $f(a) = f(b) \Rightarrow \dots$ ”.
2. Explain in one or two sentences why every function that is increasing on its whole domain is one-to-one.
3. Is every one-to-one function increasing or decreasing? If not, sketch or describe a counterexample.

Use the Horizontal Line Test. For each described graph, state whether the function is one-to-one.

4. A line with slope -4 .
5. A parabola opening upward with vertex $(2, -5)$.
6. The graph of $y = x^3 - x$.
7. The graph of $y = 2^x + 1$.

Prove one-to-one, or show it fails. If the function is one-to-one, give an algebraic proof; if not, give two inputs with the same output.

8. $f(x) = 7x + 2$
9. $f(x) = x^2 + 4x$
10. $f(x) = \frac{1}{x}, \quad x \neq 0$
11. $f(x) = \frac{x-5}{x+2}, \quad x \neq -2$
12. $f(x) = |x-3|$
13. $f(x) = x^3 + 1$
14. $f(x) = \sqrt[3]{2x-7}$

Restricting the domain.

15. Give a domain restriction on which $f(x) = (x-4)^2$ is one-to-one, and state whether f is increasing or decreasing there.
16. Give a domain restriction on which $f(x) = 9 - x^2$ is one-to-one.



Answers

Concept check.

1. f is one-to-one if, for all a, b in its domain, $f(a) = f(b) \Rightarrow a = b$.
2. If f is increasing on its whole domain and $a \neq b$, then one input is smaller, say $a < b$; increasing then gives $f(a) < f(b)$, so $f(a) \neq f(b)$. No two distinct inputs share an output, so f is one-to-one.
3. No. For example, $f(x) = \frac{1}{x}$ (domain $x \neq 0$) is one-to-one, but it is neither increasing nor decreasing on its whole domain: it decreases on $(-\infty, 0)$ and on $(0, \infty)$ separately, yet $f(-1) = -1 < 1 = f(1)$ while $-1 < 1$. A piecewise function with a jump, such as $f(x) = x$ for $x < 0$ and $f(x) = x + 1$ for $x \geq 0$, works too.
4. One-to-one (a non-horizontal line passes the test).
5. Not one-to-one. Any horizontal line above the vertex meets the parabola twice (points equally spaced on either side of $x = 2$).
6. Not one-to-one. For instance $y = x^3 - x$ gives $y = 0$ at $x = -1, 0, 1$.
7. One-to-one ($y = 2^x + 1$ is increasing on all of $\mathbb{R}$).
8. One-to-one. $7a + 2 = 7b + 2 \Rightarrow 7a = 7b \Rightarrow a = b$.
9. Not one-to-one. Vertex at $x = -2$; $f(0) = 0$ and $f(-4) = 16 - 16 = 0$, with $0 \neq -4$.
10. One-to-one. $\frac{1}{a} = \frac{1}{b} \Rightarrow b = a$.
11. One-to-one. From $\frac{a-5}{a+2} = \frac{b-5}{b+2}$, cross-multiply: $(a-5)(b+2) = (b-5)(a+2)$, i.e. $ab + 2a - 5b - 10 = ab + 2b - 5a - 10$. Cancel ab and -10 : $2a - 5b = 2b - 5a \Rightarrow 7a = 7b \Rightarrow a = b$.
12. Not one-to-one. $f(2) = 1 = f(4)$, with $2 \neq 4$.
13. One-to-one. $a^3 + 1 = b^3 + 1 \Rightarrow a^3 = b^3 \Rightarrow a = b$ (a real number has exactly one real cube root).
14. One-to-one. Cube both sides of $\sqrt[3]{2a-7} = \sqrt[3]{2b-7}$: $2a-7 = 2b-7 \Rightarrow a = b$.
15. On $[4, \infty)$ the function $f(x) = (x-4)^2$ is increasing, hence one-to-one. (On $(-\infty, 4]$ it is decreasing and also one-to-one.)
16. On $[0, \infty)$ the function $f(x) = 9 - x^2$ is decreasing, hence one-to-one. (On $(-\infty, 0]$ it is increasing and also one-to-one.)