A **rational inequality** is an inequality in which the unknown appears in the denominator of a fraction, such as

$$\frac{x+3}{x-4} \geq 2$$

When we solve an equation with fractions, we usually multiply both sides by the denominator to get rid of the fractions. With inequalities, this is dangerous. Recall that when we multiply both sides of an inequality by a positive number, the inequality sign stays the same, but when we multiply by a negative number, we must reverse the inequality sign. The trouble is that $x-4$ is positive for some values of x and negative for others. So when we multiply by $x-4$, we do not know what to do with the inequality sign.

What goes wrong if we are not careful

Suppose that we ignore the problem, and simply multiply both sides by $x-4$, keeping the inequality sign.

$$\begin{aligned}
\frac{x+3}{x-4} &\geq 2 && \text{multiply by } x-4\\
x+3 &\geq 2(x-4) && \text{distribute}\\
x+3 &\geq 2x-8 && \text{subtract } x\\
3 &\geq x-8 && \text{add } 8\\
11 &\geq x
\end{aligned}$$

This says that every number less than or equal to 11 is a solution. Let us check a few. The number $x=0$ is less than 11, but

$$\frac{0+3}{0-4} = -\frac{3}{4}, \quad \text{and} \quad -\frac{3}{4} \geq 2 \quad \text{is false.}$$

The number $x=4$ is also less than 11, but $\dfrac{4+3}{4-4} = \dfrac{7}{0}$ is undefined. So $x \leq 11$ is not the solution set. The work above is only correct for those values of x that make $x-4$ positive. We will see that the correct solution set is $4 < x \leq 11$.

Facts we will use

Suppose that A and B are expressions, and consider the fraction $\dfrac{A}{B}$.

- If $B=0$, the fraction is undefined. So a number that makes a denominator zero is never a solution of a rational inequality.
- The fraction $\dfrac{A}{B}$ is zero exactly when $A=0$ and $B \neq 0$.
- The fraction $\dfrac{A}{B}$ is positive when A and B have the same sign, and negative when A and B have opposite signs.

We will write each solution set in inequality notation and in interval notation, and graph it on the number line. For example, the solution set $4 < x \leq 11$ is the interval $(4, 11]$. The open circle at 4 shows that 4 is not a solution, and the filled circle at 11 shows that 11 is a solution.

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The coolest method

This brilliant method is from Ahyoung Kim. We would like to multiply both sides by the denominator, but we do not know whether it is positive or negative. Instead, we multiply both sides by the *square* of the denominator. A square is never negative, so we know what to do with the inequality sign.

The coolest method. To solve an inequality like $\dfrac{x+3}{x-4} \geq 2$, multiply both sides by $(x-4)^2$.

- The expression $(x-4)^2$ is positive for every $x \neq 4$, so we keep the inequality sign.
- At $x = 4$, the square is zero. But $x = 4$ makes the denominator zero, so it is not a solution anyway.

We must remember to remove it from the solution set at the end.

- The result is a quadratic inequality. We solve it by sketching a parabola.

Recall how we solve quadratic inequalities. To solve $(x+7)(x-3) > 0$, we sketch the graph of $y = (x+7)(x-3)$, and we look for the points of the graph that are above the x-axis. The x-coordinates of these points form the solution set.

Example 1. Solve the inequality $\dfrac{x+7}{x-3} > 0$.

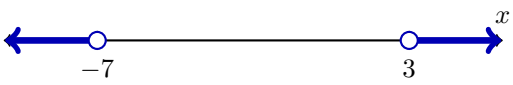
Solution: First we note that $x = 3$ is not a solution, because it makes the denominator zero. For all other values of x, the expression $(x-3)^2$ is positive, so we may multiply both sides by it and keep the inequality sign.

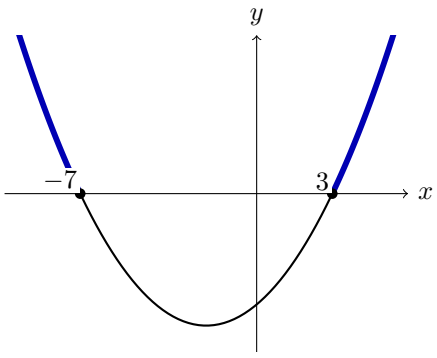
$$\begin{aligned} \frac{x+7}{x-3} &> 0 && \text{multiply by } (x-3)^2 \\ \frac{x+7}{x-3} \cdot (x-3)^2 &> 0 \cdot (x-3)^2 && \text{simplify} \\ (x+7)(x-3) &> 0 \end{aligned}$$

The graph of $y = (x+7)(x-3)$ is an upward opening parabola with x-intercepts -7 and 3. We look for the points that are above the x-axis. These are the points to the left of the first x-intercept and to the right of the second one. So the quadratic inequality is true when $x < -7$ or $x > 3$.

We have to remove $x = 3$ from the solution set, but it is not there anyway.

The solution set is $x < -7$ or $x > 3$, in interval notation, $(-\infty, -7) \cup (3, \infty)$.





$y = (x+7)(x-3)$

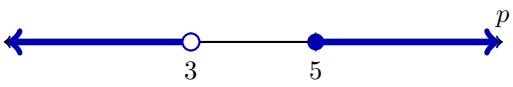
Example 2. Solve the inequality $\dfrac{p-5}{3-p} \le 0$.

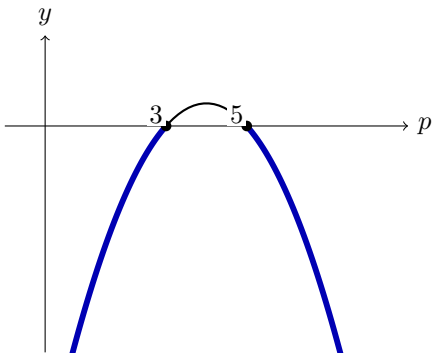
Solution: The number $p = 3$ is not a solution, because it makes the denominator zero. For all other values of p, the expression $(3-p)^2$ is positive, so we may multiply both sides by it and keep the inequality sign.

$$\begin{aligned} \frac{p-5}{3-p} &\le 0 && \text{multiply by } (3-p)^2 \\ \frac{p-5}{3-p} \cdot (3-p)^2 &\le 0 \cdot (3-p)^2 && \text{simplify} \\ (p-5)(3-p) &\le 0 \end{aligned}$$

If we multiplied out, the square term would be $-p^2$. So the graph of $y = (p-5)(3-p)$ is a downward opening parabola with intercepts 3 and 5. We look for the points that are on or below the horizontal axis: $p \le 3$ or $p \ge 5$.

Now we remove $p = 3$. The solution set is $p < 3$ or $p \ge 5$, in interval notation, $(-\infty, 3) \cup [5, \infty)$.





$y = (p-5)(3-p)$

Example 3. Solve the inequality $\dfrac{x+3}{x-4} \geq 2$.

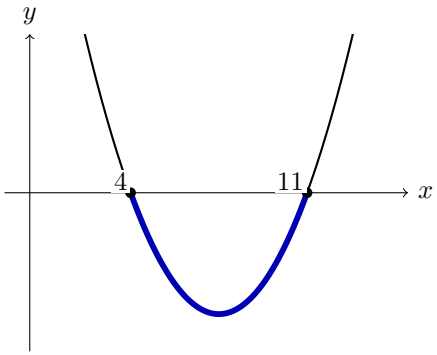
Solution: The number $x = 4$ is not a solution, because it makes the denominator zero. For all other values of x, the expression $(x-4)^2$ is positive, so we may multiply both sides by it and keep the inequality sign.

$$\begin{aligned}
\frac{x+3}{x-4} &\geq 2 && \text{multiply by } (x-4)^2 \\
\frac{x+3}{x-4} \cdot (x-4)^2 &\geq 2(x-4)^2 && \text{simplify} \\
(x+3)(x-4) &\geq 2(x-4)^2 && \text{subtract } (x+3)(x-4) \\
0 &\geq 2(x-4)^2 - (x+3)(x-4) && \text{factor out } x-4 \\
0 &\geq (x-4)(2(x-4) - (x+3)) && \text{simplify} \\
0 &\geq (x-4)(2x-8-x-3) \\
0 &\geq (x-4)(x-11)
\end{aligned}$$

It is tempting to multiply everything out, but it is better not to. When we factor out $x-4$, we get the factored form of the quadratic expression right away.

The graph of $y = (x-4)(x-11)$ is an upward opening parabola with x-intercepts 4 and 11. We look for the points that are on or below the x-axis. Their x-coordinates are between the two intercepts: $4 \leq x \leq 11$.

Now we remove $x = 4$. The solution set is $4 < x \leq 11$, in interval notation, $(4, 11]$.



$y = (x-4)(x-11)$

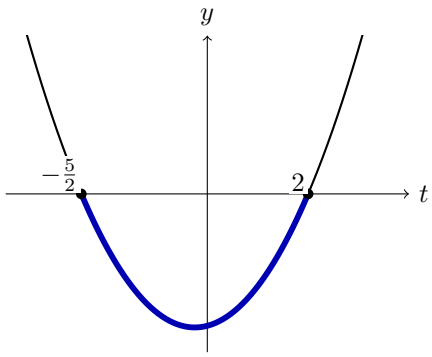
Example 4. Solve the inequality $\dfrac{5t-1}{t-2} \le 3$.

Solution: The number $t = 2$ is not a solution. For all other values of t, the expression $(t-2)^2$ is positive.

$$\begin{aligned}
\frac{5t-1}{t-2} &\le 3 && \text{multiply by } (t-2)^2 \\
(5t-1)(t-2) &\le 3(t-2)^2 && \text{subtract } 3(t-2)^2 \\
(5t-1)(t-2) - 3(t-2)^2 &\le 0 && \text{factor out } t-2 \\
(t-2)(5t-1-3(t-2)) &\le 0 && \text{simplify} \\
(t-2)(5t-1-3t+6) &\le 0 \\
(t-2)(2t+5) &\le 0
\end{aligned}$$

The graph of $y = (t-2)(2t+5)$ is an upward opening parabola. Its intercepts are 2 and $-\dfrac{5}{2}$ (where $2t+5=0$). The points on or below the horizontal axis are between the intercepts: $-\dfrac{5}{2} \le t \le 2$.

Now we remove $t = 2$. The solution set is $-\dfrac{5}{2} \le t < 2$, in interval notation, $\left[-\dfrac{5}{2}, 2\right)$.



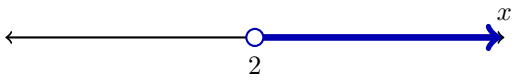
$y = (t-2)(2t+5)$

Example 5. Solve the inequality $\dfrac{3x+1}{x-2} > 3$.

Solution: The number $x = 2$ is not a solution. For all other values of x, the expression $(x-2)^2$ is positive.

$$\begin{aligned}
\frac{3x+1}{x-2} &> 3 && \text{multiply by } (x-2)^2 \\
(3x+1)(x-2) &> 3(x-2)^2 && \text{subtract } (3x+1)(x-2) \\
0 &> 3(x-2)^2 - (3x+1)(x-2) && \text{factor out } x-2 \\
0 &> (x-2)(3(x-2) - (3x+1)) && \text{simplify} \\
0 &> (x-2)(3x-6-3x-1) && \\
0 &> -7(x-2) && \text{divide by } -7\text{, reverse the sign} \\
0 &< x-2 && \text{add } 2 \\
2 &< x &&
\end{aligned}$$

This time the x-terms cancel in the second factor, and we are left with a linear inequality. There is no need for a parabola. The number $x = 2$ is not in the set $x > 2$, so there is nothing to remove.



The solution set is $x > 2$, in interval notation, $(2, \infty)$.

Practice Problems

1. Solve each of the following inequalities.

   a) $\dfrac{a-1}{a} > 0$

   b) $\dfrac{3x+6}{2x-12} \le 0$

   c) $\dfrac{-x+8}{x-2} \ge 5$

   d) $\dfrac{b+3}{5-2b} \le 4$

   e) $\dfrac{3x-1}{x} \le -1$

   f) $\dfrac{-2x+5}{x+6} > -2$

   g) $\dfrac{2}{p-1} \ge \dfrac{3}{4}$

   h) $\dfrac{2}{m+3} \le 1$

   i) $\dfrac{3}{x-2} > 1$

   j) $\dfrac{1}{x} \le 3$

   k) $\dfrac{3x}{x+5} > 2$

   l) $\dfrac{x}{x-2} > \dfrac{3}{4}$

2. a) Solve the inequality $\dfrac{x+1}{x-1} \ge 3$.

   b) Solve the inequality $\dfrac{x+1}{x-1} < 3$.

   c) Every real number makes exactly one of the statements $A \ge 3$ and $A < 3$ true. Is every real number a solution of exactly one of the two inequalities above? Explain.

3. A student solved the inequality $\dfrac{2x+5}{x-1} > 1$ by multiplying both sides by $x-1$, and got $x > -6$.

   a) Show that $x = 0$ satisfies $x > -6$, but it is not a solution of the inequality.

   b) What went wrong? Solve the inequality correctly.

4. Solve the inequality $\dfrac{3x}{x-7} < 3$. What is special about this one?

Enrichment

5. Solve each of the following inequalities.

   a) $\dfrac{2x-5}{x+2} < \dfrac{x}{4}$

   b) $\dfrac{x-1}{x+2} \ge \dfrac{x+3}{x-4}$

6. To solve $\dfrac{x+3}{x-4} \ge 2$, we could multiply both sides by $(x-4)^2$. Would it also work to multiply both sides by $(x-4)^4$? How about $(x-4)^3$? Explain.

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Answers

1. a) $a < 0$ or $a > 1$, $(-\infty, 0) \cup (1, \infty)$ b) $-2 \le x < 6$, $[-2, 6)$ c) $2 < x \le 3$, $(2, 3]$
   d) $b \le \frac{17}{9}$ or $b > \frac{5}{2}$, $\left(-\infty, \frac{17}{9}\right] \cup \left(\frac{5}{2}, \infty\right)$ e) $0 < x \le \frac{1}{4}$, $\left(0, \frac{1}{4}\right]$ f) $x > -6$, $(-6, \infty)$
   g) $1 < p \le \frac{11}{3}$, $\left(1, \frac{11}{3}\right]$ h) $m < -3$ or $m \ge -1$, $(-\infty, -3) \cup [-1, \infty)$ i) $2 < x < 5$, $(2, 5)$
   j) $x < 0$ or $x \ge \frac{1}{3}$, $(-\infty, 0) \cup \left[\frac{1}{3}, \infty\right)$ k) $x < -5$ or $x > 10$, $(-\infty, -5) \cup (10, \infty)$
   l) $x < -6$ or $x > 2$, $(-\infty, -6) \cup (2, \infty)$

2. a) $1 < x \le 2$, $(1, 2]$ b) $x < 1$ or $x > 2$, $(-\infty, 1) \cup (2, \infty)$
   c) No. The number $x = 1$ is not a solution of either inequality, because it makes the denominator zero. Every other real number is a solution of exactly one of them.

3. a) $\frac{2 \cdot 0 + 5}{0 - 1} = -5$, and $-5 > 1$ is false. b) The student did not know whether $x - 1$ was positive or negative, and kept the inequality sign. The solution set is $x < -6$ or $x > 1$, $(-\infty, -6) \cup (1, \infty)$

4. $x < 7$, $(-\infty, 7)$. The inequality is equivalent to $\frac{21}{x - 7} < 0$, so it becomes a linear inequality, and the solution set is a single ray.

Enrichment

5. a) $x > -2$, $(-2, \infty)$. Hint: the inequality is equivalent to $0 < (x + 2)\left(x^2 - 6x + 20\right)$ for $x \ne -2$, and $x^2 - 6x + 20 = (x - 3)^2 + 11$ is always positive.
   b) $x < -2$ or $-\frac{1}{5} \le x < 4$, $(-\infty, -2) \cup \left[-\frac{1}{5}, 4\right)$. Hint: the inequality is equivalent to $\frac{-10x - 2}{(x + 2)(x - 4)} \ge 0$.

6. Multiplying by $(x - 4)^4$ also works, because $(x - 4)^4$ is positive for every $x \ne 4$. Multiplying by $(x - 4)^3$ does not help: $(x - 4)^3$ has the same sign as $x - 4$, so we would again not know what to do with the inequality sign.

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