Carl Friedrich Gauss (1777-1855) is probably one of the greatest mathematicians of all time. He made major contributions to most areas within mathematics. The following story is from when Gauss was still at primary school. One day Gauss' teacher asked his class to add together all the numbers from 1 to 100, assuming that this task would occupy them for quite a while. He was shocked when young Gauss, after a few seconds thought, wrote down the answer 5050. The teacher couldn't understand how his pupil had calculated the sum so quickly in his head, but the eight year old Gauss pointed out that the problem was actually quite simple. Solution - YouTube Link

Here is the question: $1 + 2 + 3 + ... + 100 = x$

We repeat this line, but this time, backward: $100 + 99 + 98 + ... + 1 = x$

And we will add the two lines - which will be twice the desired sum.

$$\begin{array}{ccccccccccccc} 1 & + & 2 & + & 3 & + & . & . & . & + & 100 & = & x \\ 100 & + & 99 & + & 98 & + & . & . & . & + & 1 & = & x \end{array}$$

But instead of adding row 1 and then row 2, we add the numbers column by column. The sum in the first column is $1 + 100 = 101$. The sum of the second column is $2 + 99 = 101$. The sum of the third column is $3 + 98 = 101$. And so on, the sum of each column is 101 because as we step to the right, the number in the first row increase by 1 and the number in the second row decrease by 1. Thus the sum remains 101.

$$\begin{array}{ccccccccccccc} & 1 & + & 2 & + & 3 & + & . & . & . & + & 100 & = & x \\ + & 100 & + & 99 & + & 98 & + & . & . & . & + & 1 & = & x \\ \hline & 101 & + & 101 & + & 101 & + & . & . & . & + & 101 & = & 2x \end{array}$$

When we add the same number to itself repeatedly, that can be re-written as multiplication. We have 100 columns, so we added 101 to itself 100 times.

$$\begin{aligned} 100 \cdot 101 &= 2x \\ 10\,100 &= 2x \\ 5050 &= x \end{aligned}$$

This method can be applied to adding long sums, as long as the numbers increase or decrease by the same amount.

Example 1. Find each of the given sums.

a) $4 + 8 + 12 + ... + 600$ b) $3 + 6 + 9 + ... + 2019$

Solution: a) We will apply Gauss's method.

$$\begin{array}{ccccccccccccc} & 4 & + & 8 & + & 12 & + & . & . & . & + & 600 & = & x \\ + & 600 & + & 596 & + & 592 & + & . & . & . & + & 4 & = & x \\ \hline & 604 & + & 604 & + & 604 & + & . & . & . & + & 600 & = & 2x \end{array}$$

The easy question is: what is the sum in each column? Clearly 604. It takes a bit more work to figure out how many columns are there. We label the numbers in the first row as 1, 2, 3, etc

$$\begin{array}{ccccccccccccc} 1 & & 2 & & 3 & & & & & & ? & = & x \\ \downarrow & & \downarrow & & \downarrow & & & & & & \downarrow & & \\ 4 & + & 8 & + & 12 & + & . & . & . & + & 600 & = & x \end{array}$$

and see that the last number will have to get the label 150.

So, we have 150 columns and each of them adds up to 604

$$\begin{aligned} 604+604+604+...+604 &= 2x \\ 150\cdot 604 &= 2x \\ 90\,600 &= 2x \\ 45\,300 &= x \end{aligned}$$

So the sum $4+8+12+...+600=\boxed{45\,300}$.

b) $3+6+9+...+2019$ Solution - Youtube link

Example 2. Find each of the given sum

a) $31+38+45+....+423.$ b) $30+37+44+...+2018$ Solution - Youtube link

Solution: a) We will apply Gauss's method.

$$\begin{array}{rccccccccccc} & 31 & + & 38 & + & 45 & + & . & . & . & + & 423 & = & x \\ + & 423 & + & 416 & + & 409 & + & . & . & . & + & 31 & = & x \\ \hline & 454 & + & 454 & + & 454 & + & . & . & . & + & 454 & = & 2x \end{array}$$

The easy question is: what is the sum in each column? Clearly 454. It takes a bit more work to figure out how many columns are there. We label the numbers in the first row as 1, 2, 3, etc

$$\begin{array}{ccccccccccccc} 1 & & 2 & & 3 & & & & & ? & = & x \\ \downarrow & & \downarrow & & \downarrow & & & & & \downarrow & & \\ 31 & + & 38 & + & 45 & + & . & . & . & 423 & = & x \end{array}$$

We can see that as the label increase by 1, the number in the first row increase by 7. Because of this, we might suspect that the number belonging to label n has to do with $7n$. Let us try: If $n=1$, then $7\cdot 1+b=31$ gives us $b=24$. Thus the connection between the label and the number with that label is

$$M=7n+24$$

Let's try this for $n=2$. If $n=2$, then $M=7\cdot 2+24=38$.

Let's try this for $n=3$. If $n=3$, then $M=7\cdot 3+24=45$.

And so on, our formula works. So what label n belongs to $M=423$? We just need to solve the equation

$$\begin{aligned} 423 &= 7n+24 \\ 399 &= 7n \\ 57 &= n \end{aligned}$$

Thus, there are 57 columns. The rest is easy:

$$\begin{aligned} 454+454+454+...+454 &= 2x \\ 57\cdot 454 &= 2x \\ 25\,878 &= 2x \\ 12\,939 &= x \end{aligned}$$

So the sum $31+38+45+....+423=\boxed{12\,939}$

b) $30+37+44+...+2018$ Solution - Youtube link

Example 3. The first row in a theater has 31 seats in it. The second row has one more seats than the first row. The third row has one more seats than the second row. And so on, each row has one more seats than the row before. If the last row has 191 seats in it, how many seats are there in the entire theater? Solution - Youtube link

Solution: To find the number of all seats in the theater, we need to add the number of seats in each row. The number of total seats is

$$31 + 35 + 39 + ... + 191$$

We will apply Gauss's method.

$$\begin{array}{ccccccccccccc} 31 & + & 35 & + & 39 & + & . & . & . & + & 191 & = & x \\ 191 & + & 187 & + & 183 & + & . & . & . & + & 31 & = & x \end{array}$$

The easy question is: what is the sum in each column? Clearly 222. It takes a bit more work to figure out how many columns are there. We label the numbers in the first row as 1, 2, 3, etc

$$\begin{array}{ccccccccccccc} 1 & & 2 & & 3 & & & & & & ? & = & x \\ \downarrow & & \downarrow & & \downarrow & & & & & & \downarrow & & \\ 31 & + & 35 & + & 39 & + & . & . & . & + & 191 & = & x \end{array}$$

We can see that as the label increase by 1, the number in the first row increase by 4. Because of this, we might suspect that the number belonging to label n has to do with $4n$. Let us try: If $n = 1$, then $4 \cdot 1 + b = 31$ gives us $b = 27$. Thus the connection between the label and the number with that label is

$$M = 4n + 27$$

Let's try this for $n = 2$. If $n = 2$, then $M = 4 \cdot 2 + 27 = 35$.

Let's try this for $n = 3$. If $n = 3$, then $M = 4 \cdot 3 + 27 = 39$.

And so on, our formula works. So what label n belongs to $M = 191$? We just need to solve the equation

$$\begin{array}{rcll} 191 & = & 4n + 27 & \text{subtract 27} \\ 164 & = & 4n & \text{divide by 4} \\ 41 & = & n & \end{array}$$

Thus, there are 41 columns. The rest is easy:

$$\begin{array}{rcll} 222 + 222 + 222 + ... + 222 & = & 2x & \\ 41 \cdot 222 & = & 2x & \\ 9102 & = & 2x & \text{divide by 2} \\ 4551 & = & x & \end{array}$$

So the theater has $\boxed{4551}$ seats.

Example 4. Compute the sum $\dfrac{1}{1001} + \dfrac{2}{1001} + \dfrac{3}{1001} + ... + \dfrac{1000}{1001}$ Solution - Youtube link

Practice Problems

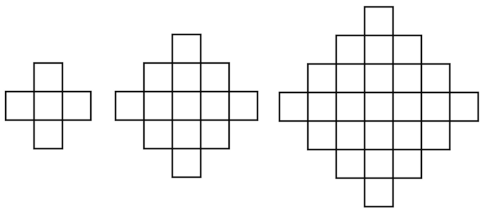
Compute each of the following sums.

1. $19 + 30 + 41 + ... + 4078$
2. $29 + 34 + 39 + ... + 374$
3. $15 + 17 + 19 + ... + 643$
4. $32 + 41 + 50 + ... + 698$
5. $11 + 17 + 23 + ... + 1325$
6. $16 + 23 + 30 + ... + 1094$
7. $20 + 27 + 34 + ... + 783$
8. $31 + 37 + 43 + ... + 1015$
9. $\frac{1}{2019} + \frac{2}{2019} + \frac{3}{2019} + ... + \frac{2018}{2019}$

10. What is the sum of all numbers shown in the square?

1	2	3	4	5	6	7	8	9	10
2	3	4	5	6	7	8	9	10	11
3	4	5	6	7	8	9	10	11	12
4	5	6	7	8	9	10	11	12	13
5	6	7	8	9	10	11	12	13	14
6	7	8	9	10	11	12	13	14	15
7	8	9	10	11	12	13	14	15	16
8	9	10	11	12	13	14	15	16	17
9	10	11	12	13	14	15	16	17	18
10	11	12	13	14	15	16	17	18	19

11. Consider the objects shown on the picture. Each are made up of unit squares. (A unit square is a square with sides 1 unit long.) The first figure consists of 5 unit squares. The second consists of 13 unit squares, the third of 25 little squares. How many unit squares make up the 100th such figure?



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Answers

1. $757\,945$
2. $14\,105$
3. $103\,635$
4. $27\,375$
5. $14\,960$
6. $86\,025$
7. $44\,165$
8. 86295
9. 1009
10. 1000
11. 20201

$$1+3+5+...+199+201+199+197+...+3+1$$
$$=201+2\,(1+3+5+...+199)=201+2\cdot\frac{100\,(200)}{2}=20\,201$$

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