

Recall the following identities. For all natural numbers n ,

$$\begin{aligned} 1 + 2 + 3 + \dots + n &= \frac{n(n+1)}{2} \\ 1^2 + 2^2 + 3^2 + \dots + n^2 &= \frac{n(n+1)(2n+1)}{6} \\ 1^3 + 2^3 + 3^3 + \dots + n^3 &= \frac{n^2(n+1)^2}{4} \end{aligned}$$



Sample Problems

Compute each of the following sums.

- $(0^2 + 2) + (1^2 + 2) + (2^2 + 2) + \dots + (20^2 + 2)$
- $(0^3 - 0^2 - 2 \cdot 0 + 1) + (1^3 - 1^2 - 2 \cdot 1 + 1) + (2^3 - 2^2 - 2 \cdot 2 + 1) + \dots + (100^3 - 100^2 - 2 \cdot 100 + 1)$
- $0.5^2 + 1^2 + 1.5^2 + 2^2 + 2.5^2 + \dots + 39.5^2 + 40^2$
- a) $3 + 6 + 9 + \dots + 300$ b) $5 + 20 + 45 + \dots + 50\,000$
- a) $2^2 + 4^2 + 6^2 + \dots + 100^2$ b) $1^2 + 3^2 + 5^2 + 7^2 + \dots + 99^2$
- $100^2 + 101^2 + 102^2 + \dots + 400^2$



Practice Problems

Compute each of the following sums.

- a) $(1^2 - 1) + (2^2 - 1) + (3^2 - 1) + \dots + (30^2 - 1)$
b) $(0^2 - 1) + (1^2 - 1) + (2^2 - 1) + \dots + (30^2 - 1)$
- $(0^2 - 6 \cdot 0 + 9) + (1^2 - 6 \cdot 1 + 9) + \dots + (28^2 - 6 \cdot 28 + 9)$
- $(1^2 - 0^2) + (2^2 - 1^2) + \dots + (100^2 - 99^2)$
- $(1^3 + 2 \cdot 1) + (2^3 + 2 \cdot 2) + \dots + (20^3 + 2 \cdot 20)$
- $(1^2 - 2 \cdot 1 + 1) + (2^2 - 2 \cdot 2 + 1) + \dots + (99^2 - 2 \cdot 99 + 1)$
- $2 + 8 + 18 + 32 + \dots + 5060$
- $4 + 16 + 36 + 64 + \dots + 10\,000$
- $5^2 + 10^2 + 15^2 + \dots + 100^2$
- $0.2^2 + 0.4^2 + 0.6^2 + \dots + 19.8^2 + 20^2$
- $0.5^2 + 1^2 + 1.5^2 + \dots + 34.5^2 + 35^2$
- $\left(\frac{1}{6}\right)^2 + \left(\frac{2}{6}\right)^2 + \left(\frac{3}{6}\right)^2 + \dots + \left(\frac{120}{6}\right)^2$
- $10^3 + 11^3 + 12^3 + \dots + 30^3$
- $(t^2 - t) = (50^2 - 50) + (51^2 - 51) + (52^2 - 52) + \dots + (100^2 - 100)$
- $1^3 + 3^3 + 5^3 + \dots + 99^3$

Sample Problems - Answers

1. 2912 2. 25 154 151 3. 43 470 4. a) 15 150 b) 1691 750 5. a) 171 700 b) 166 650 6. 21 085 050

Practice Problems - Answers

1. a) 9424 b) 9423 2. 5539 3. 10 000 4. 44 520 5. 318 549 6. 85 850 7. 171 700 8. 71 750
 9. 13 534 10. 29198.75 11. $\frac{145\,805}{9} = 16200.\bar{5}$ 12. 214 200 13. 294 100 14. 12 497 500

Sample Problems - Solutions

Recall the following identities. For all natural numbers n ,

$$\begin{aligned} 1 + 2 + 3 + \dots + n &= \frac{n(n+1)}{2} \\ 1^2 + 2^2 + 3^2 + \dots + n^2 &= \frac{n(n+1)(2n+1)}{6} \\ 1^3 + 2^3 + 3^3 + \dots + n^3 &= \frac{n^2(n+1)^2}{4} \end{aligned}$$

Compute each of the following sums.

1. $(0^2 + 2) + (1^2 + 2) + (2^2 + 2) + \dots + (20^2 + 2)$

Solution: One trick is to visually separate the constants from the numbers that change, and sum them separately

$$\begin{array}{rcl} & 0^2 & + \quad 2 \\ & 1^2 & + \quad 2 \\ S = (0^2 + 2) + (1^2 + 2) + (2^2 + 2) + \dots + (20^2 + 2) & = & \begin{array}{rcl} 2^2 & + & 2 \\ \vdots & & \vdots \\ 20^2 & + & 2 \end{array} \end{array}$$

$$S = (0^2 + 1^2 + \dots + 20^2) + (2 + 2 + 2 \dots + 2) = (1^2 + \dots + 20^2) + (2 + 2 + 2 \dots + 2)$$

What is tricky is to realize that we are adding 2 twenty-one times (and not twenty times)

$$S = \frac{20 \cdot 21 \cdot 41}{6} + 21 \cdot 2 = 2912$$

$$2. (0^3 - 0^2 - 2 \cdot 0 + 1) + (1^3 - 1^2 - 2 \cdot 1 + 1) + (2^3 - 2^2 - 2 \cdot 2 + 1) + \dots + (100^3 - 100^2 - 2 \cdot 100 + 1)$$

Solution: This problem can be made easier by organizing the terms to be added, in a rectangular form

$$\begin{array}{rcccccc}
 k = 0 & 0^3 & - & 0^2 & - & 2 \cdot 0 & + & 1 \\
 k = 1 & 1^3 & - & 1^2 & - & 2 \cdot 1 & + & 1 \\
 k = 2 & 2^3 & - & 2^2 & - & 2 \cdot 2 & + & 1 \\
 k = 3 & 3^3 & - & 3^2 & - & 2 \cdot 3 & + & 1 \\
 \vdots & \vdots & & \vdots & & \vdots & & \vdots \\
 k = 100 & 100^3 & - & 100^2 & - & 2 \cdot 100 & + & 1
 \end{array}$$

We will deal with each column separately. The first column is

$$S_1 = 0^3 + 1^3 + 2^3 + 3^3 + \dots + 100^3 = 1^3 + 2^3 + 3^3 + \dots + 100^3 = \left(\frac{100 \cdot 101}{2} \right)^2 = 25\,502\,500$$

The second column is

$$S_2 = -0^2 - 1^2 - 2^2 - 3^2 - \dots - 100^2 = -(1^2 + 2^2 + 3^2 + \dots + 100^2) = -\frac{100 \cdot 101 \cdot 201}{6} = -338\,350$$

The third column is

$$S_3 = -2 \cdot 0 - 2 \cdot 1 - 2 \cdot 2 - 2 \cdot 3 - \dots - 2 \cdot 100 = -2(1 + 2 + 3 + \dots + 100) = -2 \frac{100 \cdot 101}{2} = -10\,100$$

And the fourth column is (again, we must be careful with the $k = 0$ term that does contribute 1)

$$S_4 = 1 + 1 + 1 + \dots + 1 = 101$$

So the sum is then

$$S = S_1 + S_2 + S_3 + S_4 = 25\,502\,500 - 338\,350 - 10\,100 + 101 = \boxed{25\,154\,151}$$

$$3. 0.5^2 + 1^2 + 1.5^2 + 2^2 + 2.5^2 + \dots + 39.5^2 + 40^2$$

Solution: In case of this problem, fractions are much more helpful than decimals. Let us express this sum in terms of fractions first. Then we will bring every term to the same denominator to better see the patterns.

$$\begin{aligned}
 S &= 0.5^2 + 1^2 + 1.5^2 + 2^2 + 2.5^2 + \dots + 39.5^2 + 40^2 \\
 &= \left(\frac{1}{2}\right)^2 + 1^2 + \left(\frac{3}{2}\right)^2 + 2^2 + \left(\frac{5}{2}\right)^2 + \dots + \left(\frac{79}{2}\right)^2 + 40^2 \\
 &= \left(\frac{1}{2}\right)^2 + \left(\frac{2}{2}\right)^2 + \left(\frac{3}{2}\right)^2 + \left(\frac{4}{2}\right)^2 + \left(\frac{5}{2}\right)^2 + \dots + \left(\frac{79}{2}\right)^2 + \left(\frac{80}{2}\right)^2 \\
 &= \frac{1^2}{4} + \frac{2^2}{4} + \frac{3^2}{4} + \frac{4^2}{4} + \frac{5^2}{4} + \dots + \frac{79^2}{4} + \frac{80^2}{4} = \frac{1^2 + 2^2 + 3^2 + \dots + 80^2}{4} = \frac{1}{4} \cdot \frac{80 \cdot 81 \cdot 161}{6} = \boxed{43\,470}
 \end{aligned}$$

4. a) $3 + 6 + 9 + \dots + 300$

Solution:

$$3 + 6 + 9 + 12 + \dots + 300 = 3(1 + 2 + 3 + 4 + \dots + 100) = 3 \cdot \frac{100 \cdot 101}{2} = \boxed{15\,150}$$

b) $5 + 20 + 45 + \dots + 50\,000$

$$\begin{aligned} 5 + 20 + 45 + \dots + 50\,000 &= 5 \cdot 1 + 5 \cdot 4 + 5 \cdot 9 + \dots + 5 \cdot 10\,000 = 5 \cdot 1^2 + 5 \cdot 2^2 + 5 \cdot 3^2 + \dots + 5 \cdot 100^2 \\ &= 5(1^2 + 2^2 + 3^2 + \dots + 100^2) = 5 \cdot \frac{100 \cdot 101 \cdot 201}{6} = \boxed{1691\,750} \end{aligned}$$

5. a) $2^2 + 4^2 + 6^2 + \dots + 100^2$

Solution:

$$\begin{aligned} S &= 2^2 + 4^2 + 6^2 + \dots + 100^2 = (2 \cdot 1)^2 + (2 \cdot 2)^2 + (2 \cdot 3)^2 + \dots + (2 \cdot 50)^2 = \\ &= 4 \cdot 1^2 + 4 \cdot 2^2 + 4 \cdot 3^2 + \dots + 4 \cdot 50^2 = 4(1^2 + 2^2 + 3^2 + \dots + 50^2) = 4 \cdot \frac{50 \cdot 51 \cdot 101}{6} = \boxed{171\,700} \end{aligned}$$

b) $1^2 + 3^2 + 5^2 + 7^2 + \dots + 99^2$

Solution:

$$\begin{aligned} S &= (1^2 + 2^2 + 3^2 + \dots + 100^2) - (2^2 + 4^2 + 6^2 + \dots + 100^2) \\ &= \frac{100 \cdot 101 \cdot 201}{6} - \left((2 \cdot 1)^2 + (2 \cdot 2)^2 + (2 \cdot 3)^2 + \dots + (2 \cdot 50)^2 \right) \\ &= 338\,350 - 4(1^2 + 2^2 + 3^2 + \dots + 50^2) = 338\,350 - 171\,700 = \boxed{166\,650} \end{aligned}$$

6. $100^2 + 101^2 + 102^2 + \dots + 400^2$

Solution:

$$\begin{aligned} S &= 100^2 + 101^2 + 102^2 + \dots + 400^2 = (1^2 + 2^2 + 3^2 + \dots + 400^2) - (1^2 + 2^2 + 2^2 + \dots + 99^2) \\ &= \frac{400 \cdot 401 \cdot 801}{6} - \frac{99 \cdot 100 \cdot 199}{6} = \boxed{21\,085\,050} \end{aligned}$$