Sample Problems

1. Find the value of m so that the quadratic equation $x^2 - 8x + 6m = 2$ has exactly one solution for x.
2. Consider the equation $x^2 + 2mx + 46 = 10x + 6m$. Find all values of m for which the equation has exactly one real solution for x.
3. Find an equation of the tangent line drawn to the graph of $y = x^2 - 9x + 7$ with slope -3.
4. Find an equation of the tangent line(s) drawn to the graph of $y = \frac{1}{2}x^2 + 3x + 3$ from the point $(0, 1)$.
5. Find an equation for all tangent lines drawn to the graph of $f(x) = \frac{1}{2}x^2 - 6x + 30$ from the point $(7, 8)$.
6. Find an equation for all tangent lines drawn to the graph of $y = x^2 - 10x + 16$ from the point $(7, -14)$.

Practice Problems

1. Find the value of p so that the quadratic equation $x^2 + 10x + 3p - 8 = 0$ has exactly one solution for x.
2. Consider the parametric equation $x^2 - 6x + 49 = 2ax + 12a$. Find all values of a for which the equation has one real solution for x.
3. Find an equation of the tangent line drawn to the graph of $y = x^2 - 17x + 59$ with slope -3.
4. In each case, find an equation for all tangent lines drawn to the given graph from the point given.

   a) $y = x^2 - 8x + 3$ from $P(0, -6)$

   b) $f(x) = 7x - \frac{1}{2}x^2 - 20$ from $P(4, 8)$

   c) $y = \frac{1}{2}x^2 + 3x - 11$ from $P(2, -5)$

   d) $f(x) = \frac{1}{2}x^2 - x + 2$ from $P(3, -1)$

Sample Problems - Answers

1. 3
2. $7, -3$
3. $y = -3x - 2$
4. $y = x + 1$ and $y = 5x + 1$
5. $y = 4x - 20$ and $y = -2x + 22$
6. $y = -2x$ and $y = 10x - 84$

Practice Problems – Answers

1. 11
2. $-20,\ 2$
3. $y = -3x + 10$
4. In each case, find an equation for all tangent lines drawn to the given graph from the point given.
   a) $y = -2x - 6$ and $y = -14x - 6$
   b) $y = -x + 12$ and $y = 7x - 20$
   c) $y = 3x - 11, y = 7x - 19$
   d) $y = -x + 2$ and $y = 5x - 16$

Sample Problems – Solutions

1. Find the value of m so that the quadratic equation $x^2 - 8x + 6m = 2$ has exactly one solution for x.
Solution: We complete the square

$$\begin{aligned} x^2 - 8x + 6m &= 2 \\ x^2 - 8x + 6m - 2 &= 0 \\ \underbrace{x^2 - 8x + 16} - 16 + 6m - 2 &= 0 \\ (x-4)^2 + 6m - 18 &= 0 \end{aligned}$$

For all values of m, this expression is quadratic. If we look at this as a parabola, its vertex has

$$\begin{aligned} x - \text{coordinate:} &\quad 4 \\ y - \text{coordinate:} &\quad 6m - 18 \end{aligned}$$

for exactly one solution (or $x-$intercept), the vertex needs to be on the x-axis. In toher words, this quadratic equation has exactly one solution if the quantity added to the complete square is zero. This gives us an equation we can solve for m.

$$\begin{aligned} 6m - 18 &= 0 \\ m &= 3 \end{aligned}$$

2. Consider the equation $x^2 + 2mx + 46 = 10x + 6m$. Find all values of m for which the equation has exactly one real solution for x.
Solution: We reduce one side to zero and express the other side as a polynomial in x. (Note that each coefficient is expressed as a polynomial in m.)

$$\begin{aligned} x^2 + 2mx + 46 &= 10x + 6m \\ x^2 + 2mx - 10x + 46 - 6m &= 0 \\ x^2 + (2m - 10)x + (-6m + 46) &= 0 \end{aligned}$$

We "pretend" that we know the value of m and complete the square. Half of the linear coefficient is $m - 5$ and so the complete square we will push our expression towards is

$$(x + (m-5))^2 = x^2 + (2m - 10)x + (m-5)^2$$

and so we smuggle in $(m-5)^2$.

$$\begin{aligned} x^2 + (2m - 10)x + (-6m + 46) &= 0 \\ \underbrace{x^2 + (2m - 10)x + (m-5)^2} - (m-5)^2 + (-6m + 46) &= 0 \\ (x + (m-5))^2 - (m-5)^2 + (-6m + 46) &= 0 \\ (x + (m-5))^2 - \left(m^2 - 10m + 25\right) + (-6m + 46) &= 0 \\ (x + (m-5))^2 - m^2 + 10m - 25 - 6m + 46 &= 0 \\ (x + (m-5))^2 - m^2 + 4m + 21 &= 0 \end{aligned}$$

For all values of m, this expression is quadratic. If we look at this as a parabola, its vertex has

$$\begin{array}{ll} x-\text{coordinate:} & -(m-5) \\ y-\text{coordinate:} & -m^2+4m+21 \end{array}$$

for exactly one solution (or $x-$intercept), the vertex needs to be on the x-axis. Which is the same as

$$\begin{aligned} -m^2+4m+21 &= 0 \\ -\left(m^2-4m-21\right) &= 0 \\ -(m-7)(m+3) &= 0 \implies m_1 = 7 \qquad m_2 = -3 \end{aligned}$$

Thus the equaton will have one solution (and it will be $-(m-5) = -m+5$) when m is 7 or -3. We can check: if $m=7$, then

$$\begin{aligned} x^2+2mx+46 &= 10x+6m \qquad \text{and} \quad m=7 \\ x^2+2(7)x+46 &= 10x+6(7) \\ x^2+14x+46 &= 10x+42 \\ x^2+4x+4 &= 0 \\ (x+2)^2 &= 0 \\ x &= -2 \end{aligned}$$

If $m=-3$, then

$$\begin{aligned} x^2+2mx+46 &= 10x+6m \qquad \text{and} \quad m=-3 \\ x^2+2(-3)x+46 &= 10x+6(-3) \\ x^2-6x+46 &= 10x-18 \\ x^2+16x+64 &= 0 \\ (x+8)^2 &= 0 \\ x &= -8 \end{aligned}$$

3. Find an equation of the tangent line drawn to the graph of $y=x^2-9x+7$ with slope -3.
Solution: We are looking for the equation of a line with slope -3. Its slope-intercept form is then $y=-3x+b$. We need to find the value of b.

$$\begin{aligned} y &= -3x+b \\ y &= x^2-9x+7 \end{aligned}$$

We need to find the value(s) of b that will result in exactly one solution for the system shown above.

$$\begin{aligned} x^2-9x+7 &= -3x+b \\ x^2-6x+7-b &= 0 \\ \underbrace{x^2-6x+9}-9+7-b &= 0 \\ (x-3)^2-2-b &= 0 \end{aligned}$$

This quadratic equation has exactly one solution if the quantity added to the complete square is zero.

$$\begin{aligned} -2-b &= 0 \\ b &= -2 \end{aligned}$$

4. Find an equation of the tangent line(s) drawn to the graph of $y = \frac{1}{2}x^2 + 3x + 3$ from the point $(0,1)$.
Solution: We are looking for the equation of a line. Specifically, we will find its slope-intercept form, $y = mx + b$. Thus, we need to find the values of m and b. Clearly, $b = 1$ since the point given is the $y-$intercept.

$$\begin{aligned} y &= mx + b \qquad (0,1) \text{ is on the line} \\ 1 &= m(0) + b \\ 1 &= b \qquad \Longrightarrow \qquad y = mx + 1 \end{aligned}$$

Thus our tangent line, $y = mx + 1$. This equation represents all non-vertical lines passing through $(0,1)$. We will obtain the value(s) of m using the fact that it must be a tangent line. The common points of the line and the parabola have coordinates that are solution of the system

$$\begin{aligned} y &= mx + 1 \\ y &= \frac{1}{2}x^2 + 3x + 3 \end{aligned}$$

If we use substitution to solve the system, we obtain the equation

$$mx + 1 = \frac{1}{2}x^2 + 3x + 3$$

Once we solve this quadratic equation, we will have the $x-$coordinates of all intersection points. Our line is a tangent line if this quadratic equation has exactly one solution.

$$\begin{aligned} mx + 1 &= \frac{1}{2}x^2 + 3x + 3 \\ 0 &= \frac{1}{2}x^2 - mx + 3x + 2 \qquad \text{multiply by 2} \\ 0 &= x^2 - 2mx + 6x + 4 \\ 0 &= x^2 + (-2m + 6)x + 4 \end{aligned}$$

We complete the square: half of the linear coefficient is $-m + 3$

$$\begin{aligned} 0 &= x^2 + (-2m + 6)x + 4 \\ 0 &= \underbrace{x^2 + (-2m + 6)x + (-m + 3)^2} - (-m + 3)^2 + 4 \\ 0 &= (x + (-m + 3))^2 - (-m + 3)^2 + 4 \\ 0 &= (x - m + 3)^2 - (-m + 3)^2 + 4 \end{aligned}$$

This expression represents a parabola with vertex

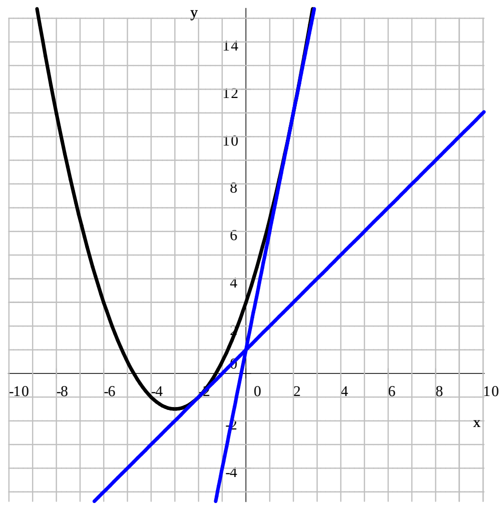
$$\begin{aligned} x - \text{coordinate:} &\qquad m - 3 \\ y - \text{coordinate:} &\qquad -(-m + 3)^2 + 4 \end{aligned}$$

There will be exactly one $x-$intercept if $-(-m + 3)^2 + 4 = 0$.

$$\begin{aligned} -(-m + 3)^2 + 4 &= 0 \\ 4 - (-m + 3)^2 &= 0 \qquad \text{difference of squares theorem} \\ 2^2 - (-m + 3)^2 &= 0 \\ (2 + (-m + 3))(2 - (-m + 3)) &= 0 \\ (-m + 5)(m - 1) &= 0 \qquad \Longrightarrow \qquad m_1 = 5 \qquad m_2 = 1 \end{aligned}$$

Thus the two tangent lines drawn from $(0,1)$ to the graph of $y=\dfrac{1}{2}x^2+3x+3$ are

$$y=x+1 \quad \text{and} \quad y=5x+1$$



5. Find an equation for all tangent lines drawn to the graph of $f(x)=\dfrac{1}{2}x^2-6x+30$ from the point $(7,8)$.
Solution: We start with the equation of all (except vertical) lines passing through $(7,8)$. These lines can all be expressed using the slope-point form and then solving for y.

$$\begin{aligned} y-8 &= m(x-7) \\ y &= mx-7m+8 \end{aligned}$$

Now the points of intersection between the parabola and line can be found by solving the system

$$\begin{aligned} y &= \frac{1}{2}x^2-6x+30 \\ y &= mx-7m+8 \end{aligned}$$

We want to find m so that this system has exactly one solution. We pretend that we know the value of m and compute for the points of intersection. Using substitution, we obtain the following quadratic (in x) equation.

$$\frac{1}{2}x^2-6x+30=mx-7m+8$$

We reduce one side to zero and arrange the terms as a polynomial in x.

$$\begin{aligned} \frac{1}{2}x^2-6x+30 &= mx-7m+8 \qquad \text{multiply by 2} \\ x^2-12x+60 &= 2mx-14m+16 \\ x^2-12x+60-2mx+14m-16 &= 0 \\ x^2+(-2m-12)x+14m+44 &= 0 \end{aligned}$$

We complete the square. Half of the linear coefficient is $\dfrac{-2m-12}{2} = -m-6$ and so we smuggle in $(-m-6)^2$ to obtain the complete square $(x+(-m-6))^2$

$$\begin{aligned}\underbrace{x^2+(-2m-12)\,x+(-m-6)^2}-(-m-6)^2+14m+44 &= 0\\ (x+(-m-6))^2-(-m-6)^2+14m+44 &= 0\end{aligned}$$

For this quadratic equation to have exactly one solution, the quantity added to the complete square must be zero.

$$(x+(-m-6))^2\underbrace{-(-m-6)^2+14m+44}_{\text{needs to be zero}}=0$$

This is a quadratic equation we can solve for m.

$$\begin{aligned}-(-m-6)^2+14m+44 &= 0\\ (-m-6)^2-14m-44 &= 0\\ m^2+12m+36-14m-44 &= 0\\ m^2-2m-8 &= 0\\ (m-4)(m+2) &= 0 \quad\Longrightarrow\quad m_1=4 \qquad m_2=-2\end{aligned}$$

We substitute these values into the equaltion of the line $y-8=m(x-7)$ to get the two equations

$$y-8=4(x-7) \quad\text{and}\quad y-8=-2(x-7)$$

We can bring these to the slope intercept form and get

$$y=4x-20 \;\text{ and }\; y=-2x+22$$

6. Find an equation for all tangent lines drawn to the graph of $y=x^2-10x+16$ from the point $(7,-14)$.
Solution: We start with the equation of all (except vertical) lines passing through $(7,-14)$. These lines can all be expressed using the slope-point form and then solving for y.

$$\begin{aligned}y+14 &= m(x-7)\\ y &= mx-7m-14\end{aligned}$$

Now the points of intersection between the parabola and line can be found by solving the system

$$\begin{aligned}y &= x^2-10x+16\\ y &= mx-7m-14\end{aligned}$$

We want to find m so that the system has exactly one solution. We pretend we know m and compute for the points of intersection.

$$\begin{aligned}x^2-10x+16 &= mx-7m-14\\ x^2-10x+16-mx+7m+14 &= 0\\ x^2-10x-mx+7m+30 &= 0\\ x^2-(m+10)\,x+7m+30 &= 0\end{aligned}$$

We complete the square.

$$\left(x - \frac{m+10}{2}\right)^2 = x^2 - (m+10)\,x + \frac{(m+10)^2}{4} \qquad \text{So we smuggle in } \frac{(m+10)^2}{4}$$

$$\begin{aligned} x^2 - (m+10)\,x + \frac{(m+10)^2}{4} - \frac{(m+10)^2}{4} + 7m + 30 &= 0 \\ \left(x - \frac{m+10}{2}\right)^2 \underbrace{- \frac{(m+10)^2}{4} + 7m + 30}_{\text{has to be zero}} &= 0 \end{aligned}$$

Now we need to find those values of m for which this equation has exactly one solution.

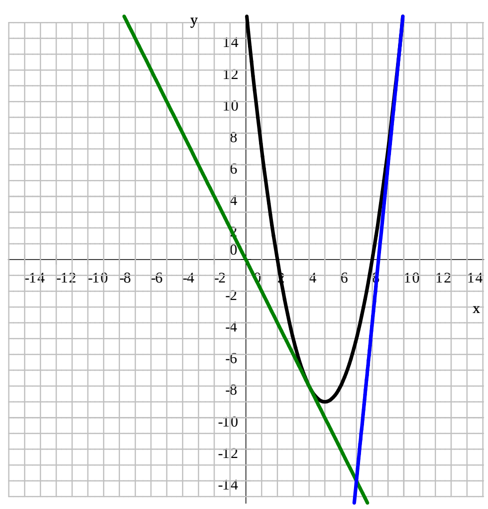
$$\begin{aligned} -\frac{(m+10)^2}{4} + 7m + 30 &= 0 \qquad \text{multiply by } -4 \\ (m+10)^2 - 28m - 120 &= 0 \\ m^2 + 20m + 100 - 28m - 120 &= 0 \\ m^2 - 8m - 20 &= 0 \\ (m+2)\,(m-10) &= 0 \qquad \Longrightarrow \qquad m_1 = -2 \qquad m_2 = 10 \end{aligned}$$

We substitute these values into the equaltion of the line $y + 14 = m\,(x - 7)$ to get the two equations

$$y + 14 = -2\,(x-7) \quad \text{and} \quad y + 14 = 10\,(x-7)$$

We can bring these to the slope intercept form and get

$$y = -2x \ \text{ (green graph) } \ \text{and} \ \ y = 10x - 84 \ \text{ (blue graph)}$$



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